# PLASTIC ANALYSIS AND DESIGN OF AXISYMMETRIC SLABS

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By
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to maiya and babuji (SHANTI SAHAI and SRI KRISHNA SAHAI) chachaji (GOPI KRISHNA SAHAY) moti and pyare

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# CERTIFICATE

This is to certify that the thesis entitled 'PLASTIC ANALYSIS AND DESIGN OF AXISYMMETRIC SLABS' by HIRA LAL is a record of work carried out under my supervision and that it has not been submitted elsewhere for a degree.

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# ABSTRACT

The main objective of a structural engineer is to design and construct a structure safe against failure. The so called "factor of safety" provided by elastic analysis does not represent the true safety factor. Plastic Analysis makes it possible to determine the actual collapse load and the factor of safety (or load factor). The basic concepts of PLASTIC ANALYSIS are briefly introduced and the literature on analytical methods of plastic analysis of plates and slabs is surveyed selectively.

The YIELD-EQUALITY METHOD of analysis (ADIDAM ~~(1972,2)~~) is discussed and used to analyse isotropically reinforced axi-symmetric circular slabs and tapered footings.

The validity of the foregoing method is tested experimentally by conducting ultimate load tests on 5 circular slabs. The test results are in close conformity with the analytical predictions and as such those analytical procedures can be adopted for design with confidence. Some more conclusions are also drawn based on the test results. Suitable recommendations for further research in this direction are also made.

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# CHAPTER I

# INTRODUCTION

## 1.1 GENERAL

STRUCTURES UNDER LOAD obviously have to be made safe against failure, and so it is vitally important to determine their maximum or ultimate strength through systematic tests and calculations. Besides strength, the structures must also have an adequate degree of rigidity, that is to say the deformations must be within some specified limits. The performance of structures within the orking load range had been the primary concern of an engineer. So in the past, and probably many years to come, structures have been and will be analysed on the basis of the theory of elasticity. This is based on the assumption that the behaviour of all structural materials within the working load range is linearly elastic.

Under this practice, structural members are proportioned and reinforced such that at working loads the maximum stresses in them are within a certain specified fraction of their corresponding failure stresses; in the case of concrete the cube crushing strength at a certain specified age (say, 28 days)

and in the case of steel the lower yield point stress obtained in a tensile test. The ratio of the failure to the permissible stress has been termed as "factor of safety" or "safety factor". Certain lower limits are also prescribed to the sizes of members to safeguard against undue deflections.

In reality none of our structural materials are truly elastic in their behaviour. Concrete is anything but elastic except at the very initial stages of loading. Steel is characterized by a very large range of plastic behaviour following the initial elastic range. So this present practice does not reflect the true material behaviour and hence lacks consistency So it would be realistic to determine the true ultimate capacity of the structures, accounting for their exact behaviour at the time of collapse. Even historically, this philosophy had been engaging the attention of researc ers. In this connection the earlier bending theory of Galilio is worth mentioning. But the later and faster developments in the field of "Theory of Elasticity" has overshadowed progress in this direction. But during the last two or three decades interest has been revived in this regard and more realistic ultimate strength theories and limit design procedures have been proposed in

order to have a philosophy of structural design consistent with the material behaviour, and reflecting the exact safety factor.

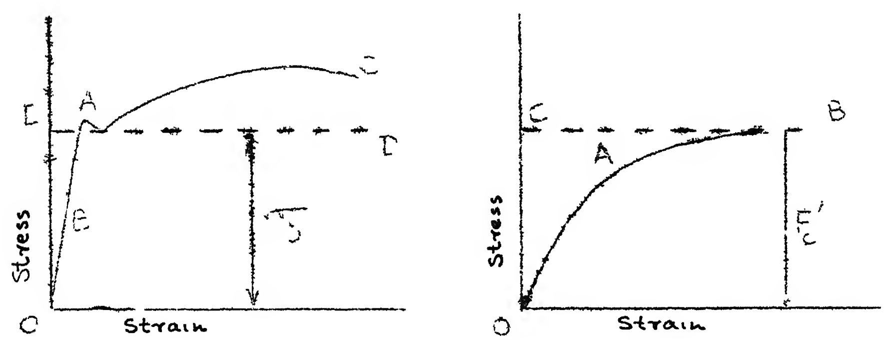
Under the new method, which is called the ultimate strength procedure the ever maximum loads likely to come upon the structure are computed from the working loads by assuming appropriate load factors. These load factors are based on statistical and behavioural considerations. The structure is then analysed and proportioned taking into account the inelastic stress-strain relationship existing at the time of collapse.

Further details and historical developments of this so called "Plastic Theory" will be elaborated in subsequent paragraphs.

## 1.2 PLASTIC ANALYSIS : BASIC CONCEPTS

Plastic analysis enables one to know the behaviour of a structure loaded into the plastic range. The actual relationship between the generalized stresses and the generalized strains of the structure are linear only at the initial stages of loading but in the later stages especially while approaching failure becomes too complex. The most exact analysis will be that which can take into account

all this complexities. The typical stress-strain diagrams are shown in fig. 1.1. One would get the exact




(a) Steel (b) Concrete

FIG. 1.1 STRESS-STRAIN CURVES

solution by considering the non-linear nature of the stress-strain curve. For the sake of simplicity these curves have been idealized as shown dotted in Fig. 1.1., that is, only the elastic-perfectly plastic (curve OBAD in Fig. 1.1.(a) or rigid-perfectly plastic(curve OEAD in Fig. 1.1.(a) and OCB in Fig 1.1(b)) behaviours are assumed.

The Plastic Limit Analysis determines the collapse loads at which the structures continue to have large deformations while the loads remain constant, in other words, unrestricted plastic flow occurs in the structures.

The limit approach to design and analysis has been adopted exclusively in this thesis, i.e. only conditions at the point of impending collapse are considered. For

certain types of structures and materials the limit approach is regarded as more rational than one based on conditions at the working load. Its advantages are that it reduces the statically indeterminate problems to statically determinate ones and does not require any previous analysis of the elasto-plastic behaviour of the system, that is, the limit load is completely independent of the loading history and all imperfections of the structure, such as settlement of supports, residual stresses etc. This system does not give an idea of the deflection under working loads and as such entails separate calculations for deflections under working loads being made if they are of primary importance in design.

Objects of Plastic Analysis are the calculation of collapse load and the determination of the true safety against collapse.

Plastic limit analysis is based on three fundamental theorems : the Lower-Bound Theorem, the Upper-Bound Theorem, (Prager and Hodge (1951,2), Prager (1959,1)) and the uniqueness theorem, (Home (1950,1), Hill (1950,2)).

### 1.2.1 Perfectly Plastic Material

No real material can be classified as perfectly

Hookean or St. Venant substance. But the Plastic theory assumes substances to be rigid perfectly plastic. In a perfectly plastic material, (St. Venant substance), indefinite strain is possible once the stress condition for yield has been reached over a large enough region for plastic flow to be geometrically possible. It does not exhibit strain hardening properties and it is capable of flowing indefinitely under zero stress rate at yield.

The generalized stresses $\underline{Q} = Q_i\ (\epsilon)$ and the corresponding generalized strains $\underline{q} = q_i\ (\epsilon)$, $i = 1,2,\ldots n$. represent the stresses or stress-resultants and strains or strain-resultants, respectively, at a point of a structure, such that the internal virtual work is given by

$$W = \sum_{i=1}^{n} Q_i \cdot q_i = \underline{Q} \cdot \underline{q} \tag{1.1}$$

In plate problems, $\underline{Q}$ represents the bending moments $(M_i)$, and $\underline{q}$ represents the curvatures $(K_i)$. In three dimensional cases, $\underline{Q}$ is the stress vector and $\underline{q}$ is the strain vector.

### 1.2.2 Physical Conditions

In the theory of elasticity, three basic requirements

are to be satisfied to analyse a structure completely, viz., equilibrium equations, compatibility conditions, and constitutive or stress-strain relations of the material used. Likewise, in the Plastic theory also, each of the three steps has to be accounted for. Equilibrium and compatibility conditions are independent of the material properties so in the plastic theory also they are unchanged, but the stress-strain relation of the material has to be considered in a different way. The stress-strain relation or the constitutive equations, known as the PHYSICAL CONDITIONS, consist of the YIELD CONDITION and the FLOW RULE. The yield condition expresses all possible combinations of the generalized stresses which produce plastic flow and is also compatible with the assumption that no plastic flow takes place under a hydrostatic system of stresses. The flow rule expresses the ratio of the plastic strain components during yielding.

## THE YIELD CONDITION

It is a function that relates the generalized stresses at yield and is given by :

$$\phi\ (\underline{Q}) = 0 \tag{1.2}$$

No yield occurs when $\phi(\underline{Q}) < 0$, yield occurs only when $\phi(\underline{Q}) = 0$, and the combinations of stresses corresponding to $\phi(\underline{Q}) > 0$ are not feasible. The point, curve or surface corresponding to equation $\phi(\underline{Q}) = 0$ is known as the yield point, the yield curve or the yield surface respectively in one-dimensional, two-dimensional, and n ($\geq 3$) ) dimensional stress space represented by the components of the stress vector $\underline{Q}$. If $n > 3$, the surface is a hyper-surface. The yield surface can be shown to be always convex (Prager (1959, 1)).

## FLOW RULE

The flow rule is assumed in such a way that strain vector $\underline{q}$ is given by,

$$q_i = \lambda \frac{\partial \phi}{\partial Q_i} \quad ; \quad i = 1, 2, \ldots n \tag{1.3}$$

where $\lambda$ is an arbitrary factor of proportionality. As the yield surface is considered to be convex and since the plastic work done is always non-negative $\lambda \geq 0$,

Equation (1.3) implies that the strain vector is normal to the yield surface given by $\phi(\underline{Q}) = 0$, hence the flow rule is sometimes called 'normality flow rule'.

Mises, using the principle of maximum plastic work, and Drucker (1951) by means of a mechanical thermodynamic postulate, substantiated the flow rule. The Mises principle of maximum plastic work will be considered herein. For a stationary value of W, if $\underline{q}$ is kept constant and Q is varied, the first variation of W is given by,

$$\delta W = \delta \underline{Q} \cdot \underline{q} = \sum_{i=1}^{n} \delta Q_i \cdot q_i = 0 \qquad (1.4)$$

Substituting $q_i$ from eq. (1.3),

$$\delta W = \sum_{i=1}^{n} \delta Q_i \cdot \lambda \frac{\partial \phi}{\partial Q_i} = \lambda \sum_{i=1}^{n} \delta Q_i \frac{\partial \phi}{\partial Q_i} = 0 \qquad (1.5)$$

The first variation of $\phi$ is given by

$$\delta \phi = \sum_{i=1}^{n} \frac{\partial \phi}{\partial Q_i} \cdot \delta Q_i = 0 \qquad (1.6)$$

From the foregoing equations it is clear that equations (1.5) and (1.6) represent the flow rule given by the equation (1.3).

If the normal to the yield surface is uniquely determined, then the point is said to be regular, otherwise it is singular. If the slope of the yield surface is

discontinuous at any point, then any convex combination of the normal vectors associated with each adjacent segment is an admissible strain vector.

### 1.2.3 YIELD CRITERIA

The most popular yield criteria are those due to Tresca, Von Mises, and Johansen. These are described in the subsequent paragraphs.

#### THE TRESCA YIELD CRITERION

In terms of principal moments (generalized stresses) in a two dimensional structure it is given by

$$|M_1| + |M_2| + |M_1 - M_2| = M_o \qquad (1.7)$$

where $M_1$ and $M_2$ are the principal moments and $M_o$ the yield moment of the plate in uniaxial bending. The yield condition and the strain vectors are shown in Fig. 1.2 (a). Tresca carried out tests on the extrusion of metals through dies and observed that the maximum shear stress reaches a definite value at yield.

#### THE MISES CRITERION

This was introduced by Von Mises on mathematical

grounds and interpreted by Hencky and independently by Huber that plastic flow occurs when the shear strain energy stored reaches a definite value. Again in terms of principal moments (generalized stresses) of a laterally loaded plate, it is given by,

$$M_1^2 - M_1 \cdot M_2 + M_2^2 = M_o^2 \qquad (1.8)$$

The yield curve and flow rule are shown in Fig. 1.2 (b).

SQUARE-YIELD CRITERION

Johansen (1943,1) intuitively adapted this criterion in the plastic analysis of reinforced concrete plates. For an isotropic slab in which there is equal amount of reinforcement at top and bottom, in terms of principal moments, the square yield criterion is given by

$$|M_i| = M_o \quad i = 1,2 \qquad (1.9)$$

The yield curve and the flow rule are shown in Fig. 1.2 (c).

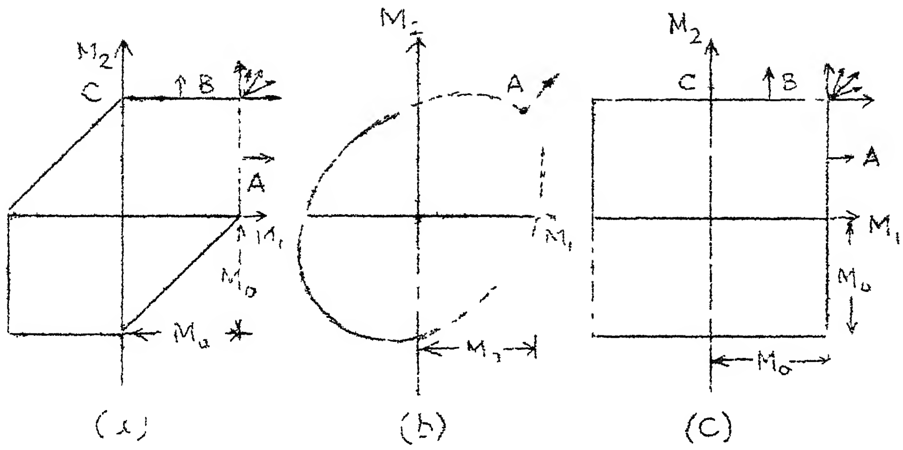



FIG. 1.2 YIELD CRITERIA

### 1.2.4 FUNDAMENTAL THEOREMS

In the limit analysis the load carrying capacity of structures is determined either by the application of the 'lower bound' or the 'upper bound' theorem or both. These theorems were first proposed by Gvozdev (1936,1) and were proved rigorously by Drucker, Prager and Greenberg (1952, 2)

SAFE (STABLE) STRESS FIELD

Any statically admissible stress field corresponding to the points within or on the yield surface is said to be a safe (stable) stress field.

KINEMATICALLY ADMISSIBLE DISPLACEMENT (VELOCITY) FIELD

Any displacement field compatible with the geometic (kinematic) boundary conditions and certain continuity conditions, is said to be a kinematically admissible displacement (velocity) field.

STATICALLY ADMISSIBLE STRESS FIELD

Any distribution of stresses satisfying the static boundary conditions and the equilibrium is said to be a statically admissible stress field.

LOAD CARRYING CAPACITY

The load carrying capacity of a structure is defined as the load at which unrestricted plastic flow occurs. This is also referred to as 'collapse load' or 'failure load' or 'limit load'.

LOWER BOUND THEOREM (Prager (1959, 1))

"In a rigid-perfectly plastic continuum, plastic flow cannot occur under loads for which a stable, statically admissible stress field can be found".

In other words, any load capacity calculated from a safe statically admissible stress field is either less

than or at the best equal to the correct load capacity.

UPPER BOUND THEOREM (Prager (1959, 1))

"In a rigid perfectly plastic continuum, plastic flow must occur under loads for which an unstable kinematically admissible velocity field can be found".

In other words, any load capacity calculated from a kinematically admissible displacement field is always more than or at the best equal to the correct load capacity.

UNIQUENESS THEOREM

For a regular yield locus, the uniqueness theorem was established by Horne (1950, 1) and Hill (1950,2). It is stated as follows :

"The load capacity is unique if derived from a safe statically admissible stress field for which a corresponding kinematically admissible displacement field exists".

## 1.3 SCOPE OF THE THESIS

The main concern of this thesis is to arbite on the YIELD EQUALITY METHOD of analysis (ADIDAM (1972,2)) for axisymmetric slabs.

Chapter I introduces very briefly the basic concepts of the Plastic Analysis.

In Chapter II, a selective literature survey of recent works on analytical methods as well as experimental tests are given with some critical comments.

In Chapter III and IV, analysis of axisymmetric slabs with varying moment capacities and footing slabs are presented. Design charts for slabs are given in Chapter III.

In Chapter V, a full report on the experiments conducted on the circular simply supported Reinforced Concrete slabs is presented. The results are discussed and suitable conclusions are drawn. Finally, suggestions for future research are made.

# CHAPTER II

# LITERATURE SURVEY

## 2.1 SELECTIVE LITERATURE SURVEY

INGERSLEV (1923,1) assuming constant bending moments along yield lines gave a method to calculate the strength of a rectangular slab.

GVOZDEV (1936,1) has given probably the first proof of the theorems of limit analysis for proportional loading. He introduced many concepts now familiar in plasticity theory, notably generalized forces and displacements, the principle of maximum plastic work and the equivalent normality condition for strain rates at yield. This concept was employed in 1952 by Prager when establishing a general theory of limit design, and Hodge has given it a central place in his text on the plastic analysis of structures. He mainly dealt with the reinforced concrete slabs.

JOHANSEN (1943,1) besides the basic introductory theory provided solutions for a large variety of practical problems in the field of plastic analysis of reinforced concrete slabs. The introduction of yield-line theory - the two alternatives :

the 'Work Method' and the 'Energy Method', stands out as the most significant contribution to the theory of perfectly plastic solids. He introduced the concept of Nodal forces-forces acting at the junction of yield lines, which was overlooked by Ingerslev. Although Johansen did not establish rigorously the yield condition for orthotropically reinforced slabs, only the 'square yield' criterion for the isotropic slabs, he proposed a method intuitively to calculate the collapse load of the slabs. This nodal force theory is also known as the 'equilibrium method' and gives only an upper bound on the collapse load since equilibrium is not necessarily satisfied throughout the domain of the slab, but only for boundaries of the plate segments separated by yield lines. A complete plastic limit Analysis (Prager and Hodge (1951)) gives identical lower and upper bounds to the correct collapse load and requires the equilibrium to be satisfied at all points of the domain.

PELL and PRAGER (1951,1) discussed the problem of load carrying capacity of circular plates, simply supported and subjected to a uniformly distributed loading. The material obeys the Mises yield criteria (yield condition and its associated flow rule). Approximate solutions were obtained using limit design theorems of Drucker, Greenberg and Prager (1951).

HOPKINS AND PRAGER (1953,1) are the first to determine the exact load carrying capacity of simply supported or clamped circular plates, subjective to rotationally symmetric loading and obeying Tresca's yield criterion by direct integration of equilibrium equations.

HOPKINS AND WANG (1954,1) obtained the exact solution for an over-hanging plate subject to a central concentrated force and a uniformly distributed loading within the support circle, by direct integration of equilibrium equation. The method was extended to plates obeying an arbitrary yield condition and subject to rotationally symmetric loading.

DRUCKER AND HOPKINS (1955,1) considered the combination of distributed and concentrated loads on circular plates with overhanging at supports and solved the problem of Hopkins and Prager (1953) as special case.

ONAT AND HAYTHORNTHWAITE (1956,1) presented an approximate analysis of rigid plastic plates obeying the Tresca yield criterion and compared the results with the test results. The load capacity after a finite deflection is estimated by assuming a kinematically admissible velocity field.

MANSFIELD (1957,1) studied the collapse mechanisms of rigid-plastic plate with a square yield criterion with particular

attention to the single concentrated load applied to a plate of arbitrary plan and with arbitrary boundary condition. He employed calculus of variations to predict the boundary shape of the fans and intended to do away with finding the lower bound solutions to find the correct collapse load. Similar solutions were obtained by Johansen (1943) by using nodal force theory.

SCHUMANN (1958,1)extended the work of Hopkins and Prager (1953) on the load carrying capacity of circular plates obeying Tresca's yield criterion to non-symmetric cases, using the same basic equations as Hopkins (1955), and determined the limit load of a plate arbitrarily shaped and subjected to a concentrated force anywhere in the plate.

CRAWFORD (1962,1) examined the principles used in limit analysis of plates and illustrated them by an example. The yield line theory is considered in terms of limit analysis and is shown to give an upper bound on the collapse load.

KEMP (1962,2) obtained a lower bound solution to the collapse load of an orthotropically reinforced rectangular-slabs, simply supported on all sides by distributing the bending and twisting moments to satisfy equilibrium and showed that the yield line pattern so obtained is different from that of upper bound solution.

NIELSON (1964,1) formulated yield condition for both isotropic and orthotropic slabs based on the characteristics of concrete and the reinforcement and presented a number of upper and lower bound solutions for different types of slabs. Nielson showed that the "Work" and "Equilibrium" methods are one and the same, by proving that the introduction of nodal forces is equivalent to the differentiation of the work equation in the work method.

PHILLIPS AND SIERKOWSKI (1964,2) introduced the concept of a family of loading surfaces that are distinct from the yield surface. In the region of stress space occupied by a family of loading surface, reloading produces plastic strains while unloading produces elastic strains only.

KEMP (1965,1) showed that the true yield criterion is a "normal" moment criterion while the tangential and the twisting moments on the yield line may vary. He noted that only the normal moment contributes to the dissipation of the energy and concluded that the yield line theory was built on the basis of normal moment yield criterion.

GURFINKEL (1965,2) tried several yield line pattern to find a lower limit of the upper bound and concluded that the yield pattern used by Crawford (1962,1) is the correct one.

PARKHILL (1966,1) described a lower bound solution for a simply supported square slab subjected to a uniform load and established the uniqueness of the load value. He also indicated the applicability of the procedure to slabs of varying geometry and loading. He critically reviewed the post-elastic bending analysis of slabs, and concluded that the elastic plastic model is more appropriate than the rigid plastic model for under reinforced slabs.

JANAS (1967,1) considered the complete set of generalized stresses and strain rates and discussed the kinematical compatibility problems in yield line theory and the statical restrictions applied in the upper bound technique. Both Johansen's classical theory and the theory taking membrane forces into account were considered.

MASSONNET (1967,2) obtained complete analytical solutions for the exact limit load of isotropic and orthotropic slabs using Hopkins (1957) and Schumann's (1958) general results, developed a criterion for the kinematically admissible mechanism and the statically admissible stress field, and illustrated the theory with various examples.

SAVE (1967,3) showed that the consistent limit-analysis theory for reinforced concrete slabs could include the yield-line theory. It is shown that the yield-line theory falls within the framework of the consistent limit-analysis.

KEMP (1967,4) obtained an upper bound solution for the collapse load of a simply supported, isotropically reinforced square slab subject to a uniformly distributed load taking membrane effects into account, in the same way as Wood (1961) for circular slabs.

LENSCHOW AND SOZEN (1967,5) developed a general yield criterion for reinforced concrete slabs subjected to a combination of flexural and twisting moments and compared with the experimental results.

MORLEY (1967,6) extended yield line theory of reinforced concrete slabs to allow for the effects of large deflections and membrane forces and illustrated the theory with numerous examples of slabs of various shapes and boundary conditions.

SAWCZUK (1967,7) studied the general tensorial relations for two dimensional continuum to find the form of a constitutive equation for plates at collapse. These laws are formulated in terms of generalised stress resultants and the corresponding displacements. The equation derived is based on the condition that it is homogeneous of zero degree in time. The yield locus and the general form of flow rule are derived. Drucker's postulate (1951) is shown to be a special case.

PRINCE AND KEMP (1968,1) developed a yield criterion for Isotropically reinforced concrete slabs based upon strain compatibility requirements across the crack along a yield line. A generalized yield criterion incorporating the expression for normal moment of resistence was presented. The square yield criterion was shown to be a lower bound of the generalized yield criterion. Experimental results show good agreement with the theory.

SAWCRUCK AND HODGE (1968,2) investigated the relationship between the yield line theory and the limits analysis and formulated a general method for finding the yield point load of simply supported isotropic slabs subjected to single point loads.

WOOD (1968,3) discussed some controversial topics in the plastic theory of structures as the yield criterion to be applied in the design of slabs, the falling moment characteristics of beams with increased curvatures and the relationship between the yield-line theory and limit analysis. It is shown that the limit analysis cannot strictly encompass the yield line theory, and the advantages of elastic analysis are highlighted.

WOOD AND ARMER (1968,5) critically examined the "strip Method" of Arne Hillerborg for the ultimate load design

of reinforced concrete slabs, and have shown that the original method is remarkably simple to apply, whereas the later developments are too complicated to be readily acceptable. They proposed modifications for the sake of simplicity and veracity, and shew that the method as applied by Hillerborg is not necessarily a "Lower-bound" solution as he intended.

ARMER (1968,6) examined the behaviour of rectangular reinforced concrete slabs with various support conditions, both at working and ultimate load levels. The slabs tested were designed by a modified version of Hillerborg's strip method.

HAYES (1968,7) review the yield line analysis methods, allowing for membrane action, due to Taylor (1965), Kemp (1967) and Sawczuck and Winnichi (1965) and outlined their limitations. Plastic analysis based on equilibrium was proposed and the method was compared with other methods and experimental results.

ROZVANY et al (1969,1) introduced a method of limit analysis for axisymmetric slabs and shells which has advantages over the usual upper bound (yield line) methods. Instead of finding the worst failure mechanism, it is only necessary to find a stress field that satisfies the equilibrium and certain yield equalities.

A comprehensive set of solutions is given for

axisymmetric slabs. It is shown that some solutions in the literature derived on the basis of yield line methods are erroneous.

HADDOW (1969,2) obtained exact yield-point loading curves relating the intensities of uniformly distributed transverse load and in-plane tensile force at the outer edge for simply supported circular plates obeying Tresca's yield criterion.

SAWCZUK (1969,3) developed a method of finding complete limit analysis solutions for orthotropic plates, just as Sawczuk and Hodge (1968) did for isotropic plates and illustrated with numerous example. The difference between the isotropic and orthoprotic yield point loads were pointed out.

WOOD (1969,4) discussed the essential differences between yield-line theory and limit analysis, examined the possibility of non-existence of exact solutions in limit analysis, suggested that Johansen's yield-line theory should not be considered as an upper bound, and that the nodal-force theory should not be considered as an upper-bound, and that the nodal-force theory could well find a place in other branches of science.

MUSPRATT (1969,5) described lower bound plastic analysis of axisymmetric steel footing plates and extended the procedure

to concrete footing slabs taking into account the earth pressure variations, plastic moment interaction, and discrete reinforcing effects. He gave experimental verification of some of the conclusions arrived at.

BRAESTRUP (1970,1) strongly refuted the statement of Wood (1965,68) and Jones and Wood (1967) that yield-line theory and limit analysis are inconsistent, and demonstrate that both the 'normal moment' criterion of the modern yield-line theory and the 'stepped' criterion of the classical theory correspond to a so called upper yield surface, which satisfies the requirements of limit analysis. It is also shown that the successive refinement of yield-line pattern does not converge to the yield load predicted by limit - analysis, when the actual yield surface of the plate is different from the upper yield surface. The yield surface of an arbitrarily reinforced plate is derived.

ROZVANY AND MELCHERS (1970,3) presented a new approach to the direct design of axisymmetric slabs. They derived a set of general rules for the form of optimal solutions and gave a comprehensive set of least reinforcement solution. Examples were given to illustrate the method showing that this approach results in a considerable saving in material.

CLYDE (1972,1) presented additional conditions for calculating the limit load on slabs.

ADIDAM (1972,2) presented an exhaustive literature survey and discussed a new method of plastic limit analysis termed "the yield equality method" to analyse a variety of axi-symmetric reinforced concrete slabs with different layouts of reinforcement. Design charts were also presented. ADIDAM used this method to design and study the post-yield behaviour of footing slabs subjected to various pressure distributions. He conducted tests on fifteen circular slabs with different boundary and loading conditions to prove the validity of the foregoing method.

## 2.2 CONCLUDING REMARKS

From the foregoing survey it is seen that much of the earlier work carried out had been mostly on upper bound solutions, provided by the Yield Line Theory and refinement of the same. They all generally pertained to under reinforced concrete slabs having uniform reinforcements. In practice the slabs are not reinforced in such a fashion.

On the other hand the work on establishing lower bound solutions are not many and the few works carried out are also recent ones. Hillerberg's strip method seems

to be the first attempt in having different reinforcements at different parts of the slabs. But it does not however make any attempt at optimizing the reinforcements. Considering all these, the works carried out at Monash University by Rozvany and his associates appear to be well directed in optimizing the reinforcements while meeting the stringent requirements of limit analysis. So it is felt that any future research will be worth the effort if it could arrive at optimal patterns of reinforcement while satisfying the limit theorems of Plastic Analysis.

# CHAPTER III

# ANALYSIS OF AXISYMMETRIC SLABS

## 3.1 YIELD EQUALITY METHOD

In 1969 Rozvany introduced a method of limit analysis for axisymmetric slabs and shells which has advantages over the usual upper bound (yield line) methods. Instead of finding the worst failure mechanism, it is only necessary to find a stress field that satisfies the equilibrium and certain yield equalities.

A theorem proposed and proved by Rozvany (1970,4) provides a unified approach for the determination of collapse loads for all types of collapse mechanisms, whether partial or total. It is based on the lower bound theorem of limit analysis. In an axisymmetric plate with a rectangular yield criterion, let the circumferential plastic moment capacity be $\bar{M}_\theta$ for positive bending and $\alpha\,\bar{M}_\theta$ for negative bending and the radial yield moments $\bar{M}_r$ and $\beta\,\bar{M}_r$. In the theorem that follows, p (r) is the axially symmetric load. $\bar{M}_\theta$, $\bar{M}_r$, $\alpha$ and $\beta$ are specified functions of radius $r$ which are piecewise continuous and bounded, and the collapse

load is $\lambda$ p (r), where $\lambda$ is the highest statically admissible multiplier.

THEOREM

"The correct load capacity (limit load or collapse load) $\lambda$ p (r) of an axi-symmetric slab is always associated with at least one piecewise continuous safe statically admissible moment field such that $M_\theta = \bar{M}_\theta$ or $M_\theta = -\alpha \bar{M}_\theta$ throughout the slab".

The moment field is said to be statically admissible if it satisfies the equilibrium equation,

$$(r. M_r)'' - M_\theta' = - p (r). r \tag{3.1}$$

where prime denotes the differentiation with respect to radius r, and it is called safe if it satisfied the yield inequalities:

$$\begin{aligned} -\beta \bar{M}_r < M_r < \bar{M}_r \\ -\alpha \bar{M}_\theta < M_\theta < \bar{M}_\theta \end{aligned} \tag{3.2}$$

This theorem by itself does not constitute a sufficient condition for the correct load capacity, unless a corresponding kinematically admissible mechanism exists. The

existence of such a mechanism can be verified by inspection of the moment diagram. In the case of partial collapse, in rigid regions of the slab an infinite number of safe statically admissible moment fields can be determined with the aid of this theorem. All the assumptions of the limit analysis of rigid ideal plastic plates are valid. Rozvany gave a proof for the case $\alpha = \beta = 1$, and Rozvany, Chamet, Adidam and Melchers (1969,1) presented a general proof for any value of $\alpha$.

The general procedure is outlined in the next article.

### 3.1.1 Procedure : Yield Equality Method

(A) Select the order of regions where $M_\theta = \bar{M}_\theta$ or $M_\theta = - \alpha \bar{M}_\theta$ (cf., Theorem 3.1).

(B) Calculate the corresponding radial moments from the equilibrium equation (3.1.1) in any region i. In the particular case of $\bar{M}_\theta$ = constant $\alpha \bar{M}_\theta$ = constant, $M'_\theta = 0$ and

$$M_r = 1/r \int_A^r \int_A^{\bar{r}} - \bar{\bar{r}}\, p_1\, (\bar{\bar{r}}).\, d\bar{\bar{r}}.\, d\bar{r} \qquad (3.3)$$

where A is the radius of the inner boundary of the slab, $\bar{r}$ and $\bar{\bar{r}}$ are dummy variables representing radii.

If the load is uniformly distributed,

$$p(r) = p, \text{ then}$$

$$M_{r1} = \frac{-pr^2}{6} + \frac{C_{11}}{r} + C_{21} \tag{3.4}$$

where, $C_{11}$ and $C_{21}$ are constants of integration.

(C) Determine the constants of integration $C_{11}$ and $C_{21}$ and the load capacity from the boundary, continuity and yield conditions.

a) Boundary Conditions

(i) In a circular slab at the axis of symmetry, except under a concentrated load, $M_r = M_\theta$ (3.5)

(ii) At free edges and simple discontinuous supports,

$$M_r = 0 \tag{3.6}$$

(iii) At free edges with no concentrated edge loads, the radial shear $Q_r$ is determined by integrating the equilibrium equation,

$$(r\, M_r)' - M_\theta = r\, Q_r = 0 \tag{3.7}$$

b) Conditions of Continuity

These refer to the continuity of radial moments and shear across a region boundary. The boundary between two adjacent regions $i$ and $(i+1)$ is termed the region boundary and the radius of the region boundary is denoted by $r_i$.

At $r = r_i$

i) Moment continuity

$$M_{ri} = M_{r\,(i+1)} \tag{3.8}$$

ii) Shear Continuity

$$(rM_{ri})' - M_{\theta i} = (rM_{r\,(i+1)})' - M_{\theta(i+1)} + Q_i \tag{3.9}$$

where $Q_i$ is the intensity of a concentrated line load, if any, acting along the region boundary.

c) Yield Conditions

Sometimes it is necessary to satisfy the radial yield equalities to obtain a unique moment field, at a finite number of points along a radius of slab.

$$M_r = \bar{M}_r \quad \text{or} \quad M_r = -\beta\,\bar{M}_r \tag{3.10}$$

In general, the maximum number of such points is :

$$\nu = \eta_e + \min(\mu_1, \mu_2) \tag{3.11}$$

where $\eta_e$ is the degree of external in determinacy (i.e. the number of reactive components that can be assigned arbitrarily in a statically admissible stress system), $\mu_1$ is the number of regions with $M_\theta = -\alpha \bar{M}_\theta$ and $\mu_2$ is the number of regions with $M_\theta = \bar{M}_\theta$.

(D) Check the sufficiency of the proposed moment field by determining whether a kinematically admissible mechanism exists.

A number of axisymmetric problems with a variety of loading and boundary conditions has been solved using this method by Rozvany and Adidam ((1969, 1) & (1972,2)). Two examples are presented here to illustrate the method. DESIGN CHARTS for the circular slabs with varying moment capacities are also given.

## 3.2 EXAMPLES

### 3.2.1 Circular Slab With Partial U.D.L.

$M_\theta = \bar{M}_\theta$ throughout the slab.

Two regions selected as in the Fig. 3.1

Corresponding radial moments are obtained by the equilibrium equation (3.1).

Or, from equation (3.4) :

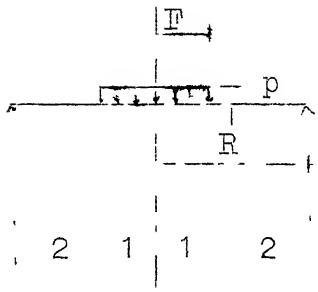



FIG. 3.1

CIRCULAR SIMPLY SUPPORTED SLAB

$$M_{r1} = \frac{-pr^2}{6} + \frac{C_{11}}{r} + C_{21} \quad ; F \geqq r \geqq 0 \tag{3.12}$$

$$M_{r2} = \frac{C_{12}}{r} + C_{22} \quad ; R \geqq r \geqq F \tag{3.13}$$

Boundary Conditions :

$$M_r = M_\theta = \bar{M}_\theta \quad \text{at } r = 0 \tag{3.14}$$

$$M_r = 0 \quad \text{at } r = R \tag{3.15}$$

Continuity Conditions :

Moment Continuity, $M_{r1} = M_{r2}$ at $r = F$ (3.16)

Shear Continuity, $(rM_{r1})' = (rM_{r2})'$ at $r = F$ (3.17)

Yield Condition :

Number of point where radial yield equality to be satisfied

by equation (3.11) comes to zero.

Evaluating $C_{11}$, $C_{21}$, $C_{12}$, and $C_{22}$ from the four equations (3.14), (3.15), (3.16) and (3.17) $\bar{M}_\theta$ is obtained,

$$\frac{\bar{M}_\theta}{p\ R^2} = \frac{f^2}{6}\ (3-2f) \tag{3.18}$$

and

$$M_{r1} = \frac{-pr^2}{6} + \frac{pF^2}{6R}\ (3R-2F) \quad ;\ F \geqq r \geqq 0 \tag{3.19}$$

$$M_{r2} = \frac{pF^3}{3}\ (1/r\ -\ 1/R) \quad ;\ R \geqq r \geqq F \tag{3.20}$$

### 3.2.2 Circular Slab With Partial Uniformly Distributed Load and Varying Moment Capacity

Consider an isotropically reinforced circular slab subject to a central partial uniformly distributed load p, and the yield moments $M_1$ and $M_2$ in regions $0 \leqq r \leqq A$ and $A \leqq r \leqq R$ respectively. The negative and positive moment capacities are equal in magnitude.

The simplest solution as per the theorem (3.1) is a circumferential moment field in which,

Case (i) $F \leq A$

$$M_{\theta 1} = M_1 \quad ; \quad 0 \leq r \leq F \tag{3.21}$$

$$M_{\theta 2} = M_1 \quad ; \quad F \leq r \leq A \tag{3.22}$$

$$M_{\theta 3} = M_2 \quad ; \quad A \leq r \leq R \tag{3.23}$$

The corresponding radial moment field :

$$M_{r1} = \frac{-pr^2}{6} + \frac{C_{11}}{r} + C_{21} \quad ; \quad 0 \leq r \leq F \tag{3.24}$$

$$M_{r2} = \frac{C_{12}}{r} + C_{22} \quad ; \quad F \leq r \leq A \tag{3.25}$$

$$M_{r3} = \frac{C_{13}}{r} + C_{23} \quad ; \quad A \leq r \leq R \tag{3.26}$$

Boundary and Continuity Conditions :

$$M_{r1} = M_1 \quad \text{at} \quad r = 0 \tag{3.27}$$

$$M_{r3} = 0 \quad \text{at} \quad r = R \tag{3.28}$$

$$M_{r1} = M_{r2} \quad \text{at} \quad r = F \tag{3.29}$$

$$M_{r2} = M_{r3} \quad \text{at} \quad r = A \tag{3.30}$$

$$(rM_{r1})' - M_1 = (rM_{r2})' - M_1 \text{ at } r = F \qquad (3.31)$$

$$(rM_{r2})' - M_1 = (rM_{r3})' - M_2 \text{ at } r = A \qquad (3.32)$$

The six unknowns $M_1$, $C_{21}$, $C_{12}$, $C_{23}$, $C_{22}$, $C_{13}$ are determined from six equations (3.27) to (3.32).

$$M_1/pR^2 = \frac{f^2(3-2f)b}{6(1+a(b-1))} \qquad (3.33)$$

$$M_{r1} = \frac{-pr^2}{6} + \frac{bpR^2(f^2(3-2f))}{6(1+a(b-1))} \; ; \; 0 \leqq r \leqq F \qquad (3.34)$$

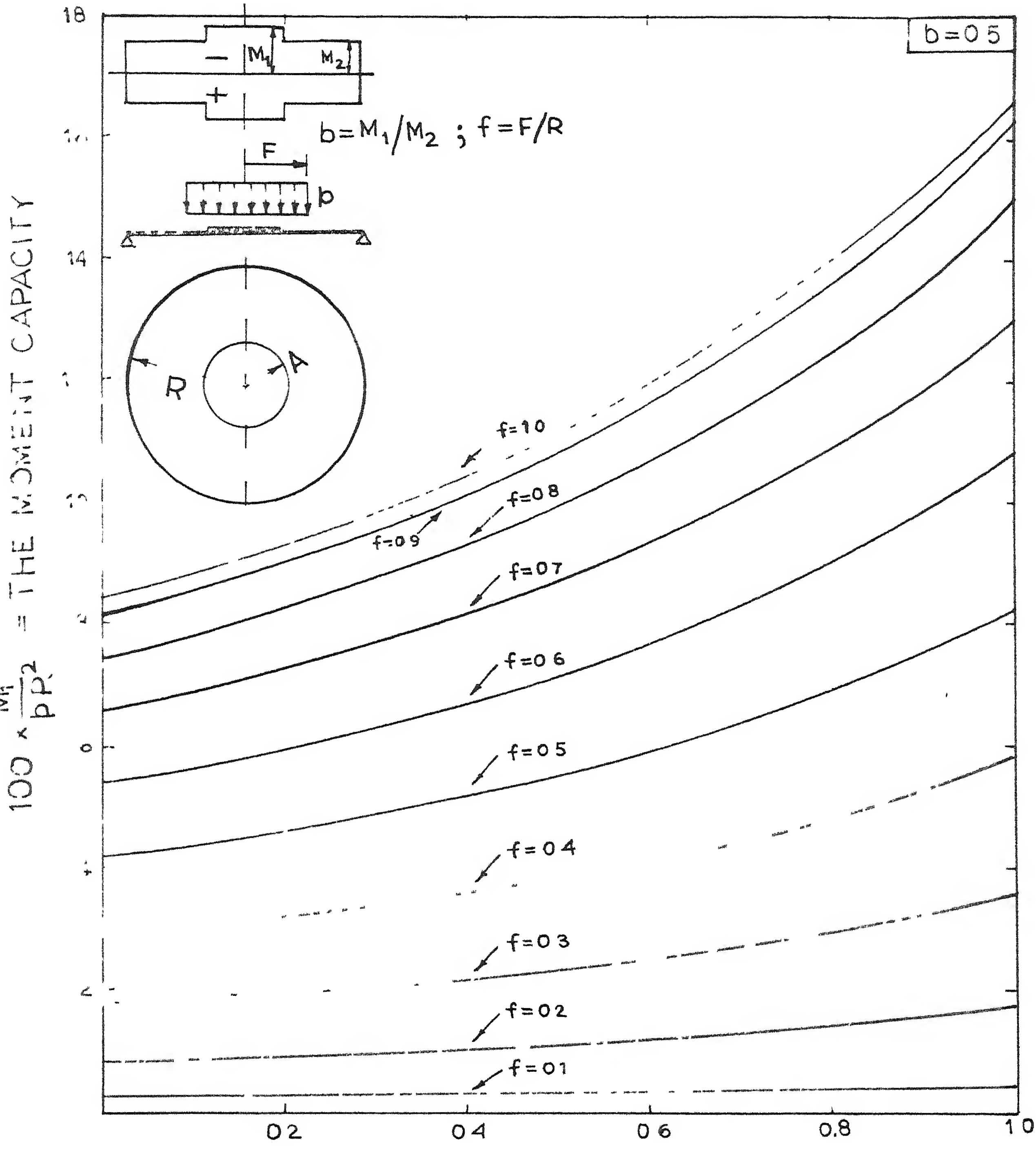
$$M_{r2} = \frac{pF^3}{3r} + \frac{bpR^2(f^2(3-2f))}{6(1+a(b-1))} - \frac{pF^2}{2} \; ; \; F \leqq r \leqq A \qquad (3.35)$$

$$M_{r3} = \frac{pR^2a}{(1-a)r}\left[\frac{f^3R}{3a} + \frac{b(f^2(3-2f))}{6(1+a(b-1))} - \frac{f^2}{2}\right] + \frac{pR^2(f^2(3-2f))}{6(1+a(b-1))} - \frac{pR^2F^2}{2} \; ; \; A \leqq r \leqq R \qquad (3.36)$$

The design charts for the values of b ( $= \frac{M_1}{M_2}$) 0.5, 1.0, 1.5 and 2.0 and f( $=\frac{F}{R}$) 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0 are given in figures (3A. 1) to (3A.4).

Case (ii) $F \geqq A$

Proceeding as before the same expression as equation (3.33) is obtained for $\frac{M_1}{pR^2}$.




$a = \frac{A}{R}$

Fig.3A.1 Design Charts for Circular Slabs of varying Moment Capacities .

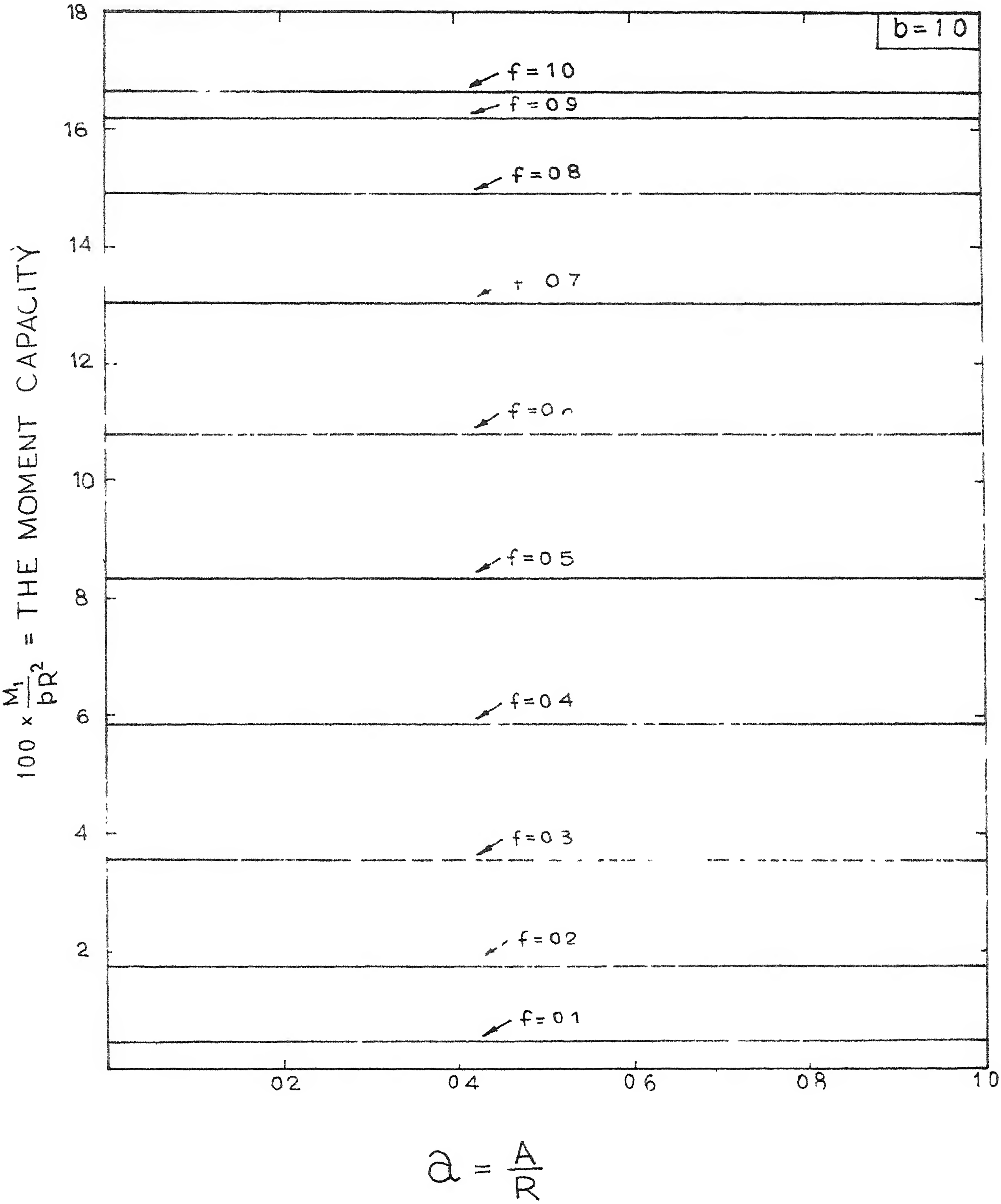



Fig 3A 2 Design Charts for Circular Slabs of varying Moment Capacities.

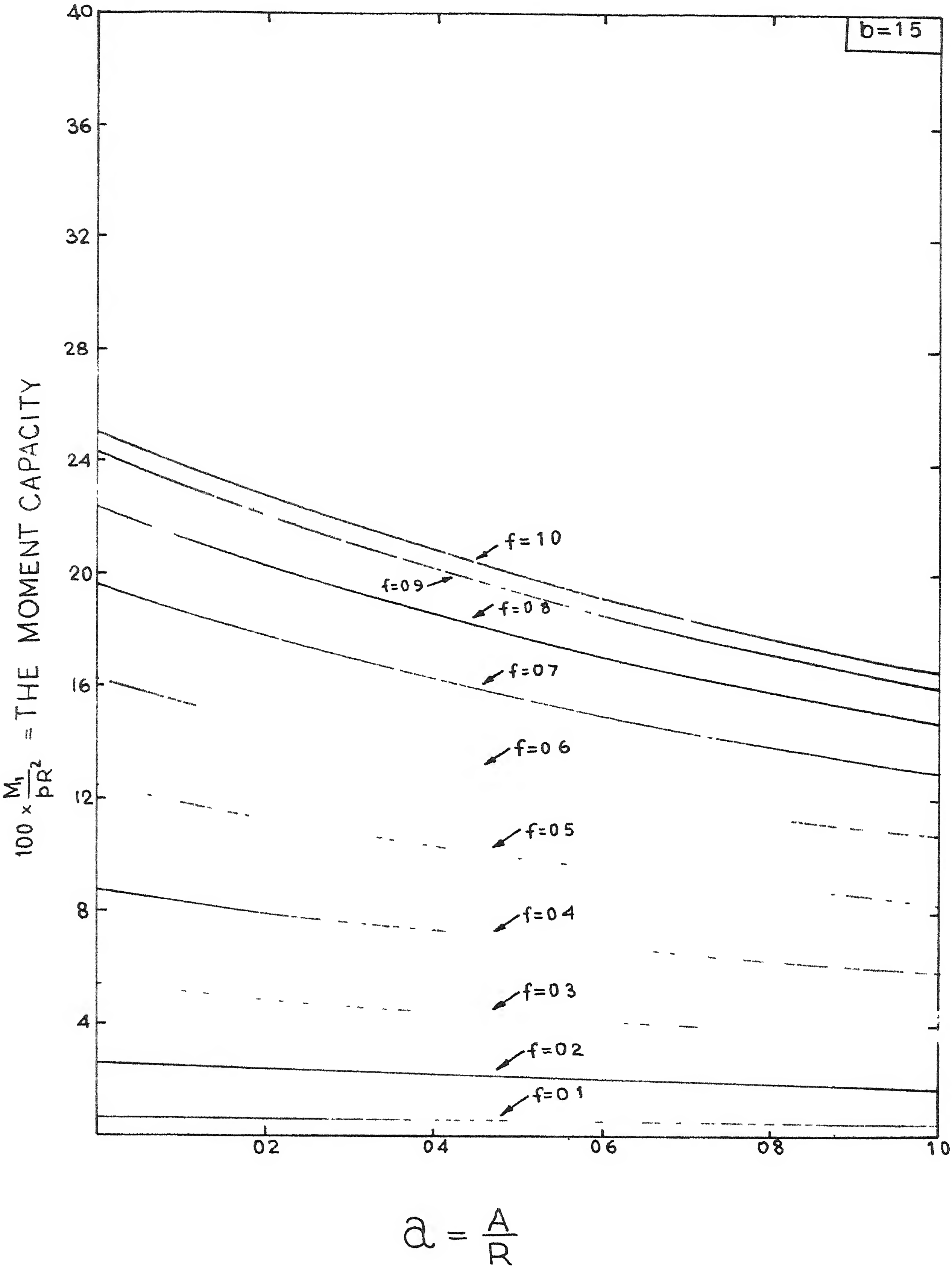



Fig. 3A.3 Design Charts for Circular Slabs of varying Moment Capacities.

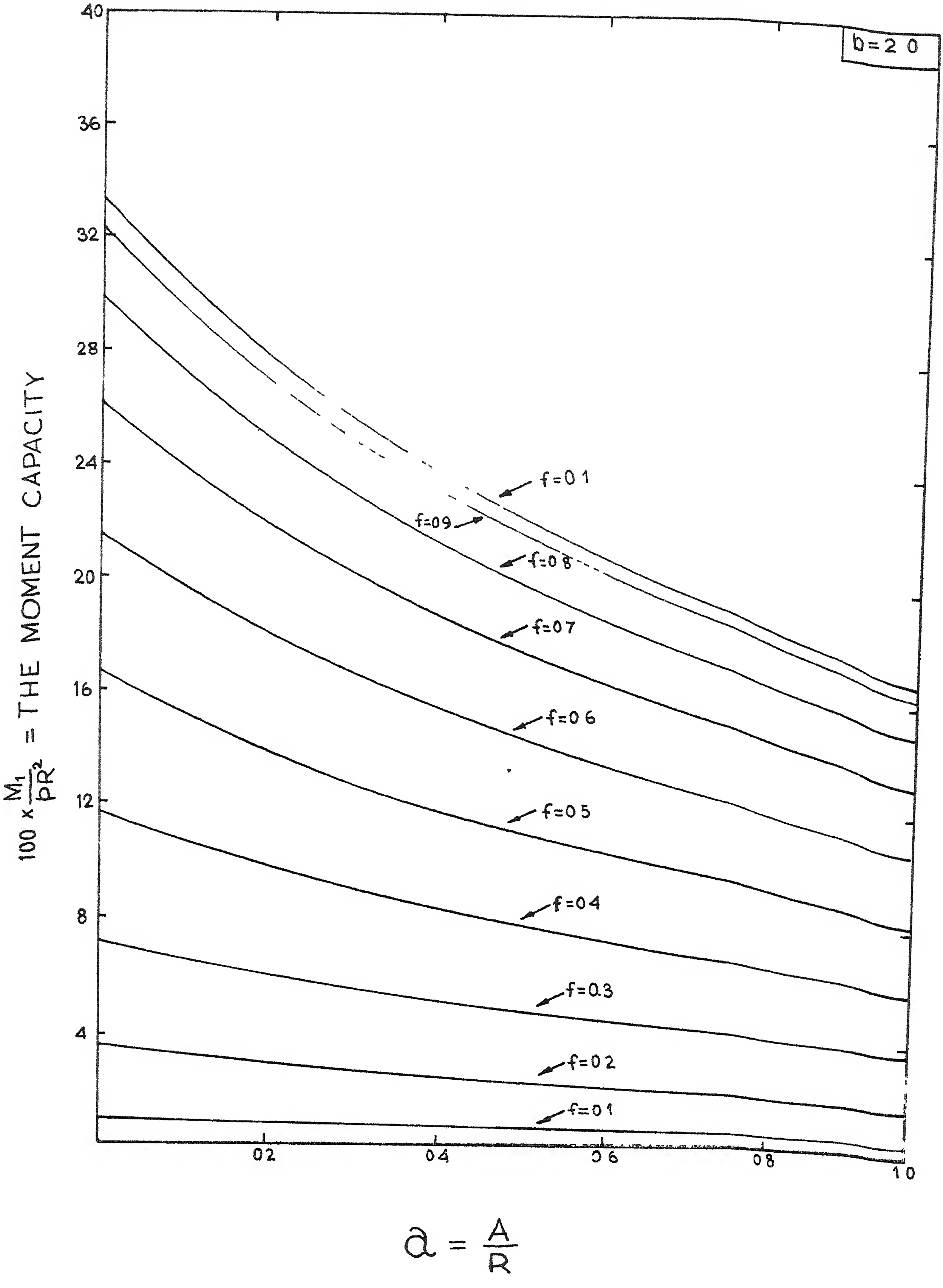



Fig. 3A 4 Design Charts for Circular Slabs of varying Moment Capacities.

# CHAPTER IV

# ANALYSIS OF AXISYMMETRIC FOOTING SLAB

## 4.1 GENERAL

In this chapter the solutions of a tapered circular footing slab are presented as per the "Yield Equality Method" only the uniform pressure distribution considered here in.

## 4.2 LIMIT DESIGN OF FOOTING SLABS

### 4.2.1 Soil Pressure Distributions

The distribution of soil pressure below footing slabs is dependent on the type of soil, the flexibility or rigidity of the footing and the eccentricity (if any) of the loading. In particular, the distribution of soil pressure below a footing on granular soil is very different from the one founded on cohesive soil. Hence the first assumption the structural designer must make is the general form of distribution of soil pressures. The most commonly adopted pressure distribution is the uniform pressure distribution. Let Q be the total load acting on a column footing of radius R. The net soil pressure

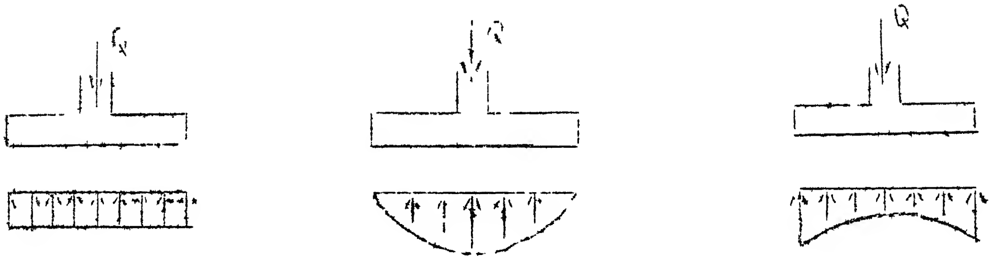
distributions for various soils is usually assumed as follows :

for uniform pressure $$p(r) = -\frac{Q}{\pi R^2}$$

for granular soils $$p(r) = -\frac{2Q}{\pi R^2}\left(1-\frac{r^2}{R^2}\right) \qquad (4.1)$$

for cohesive soil $$p(r) = -\,2/3\left(\frac{Q}{\pi R^2}\right)\left(1+\frac{r^2}{R^2}\right)$$

where r is any radius from the axis of symmetry.




$$p(r) = \frac{Q}{\pi R^2}$$

Uniform Soil Pressure

$$p(r) = \frac{2Q}{\pi R^2}\left(1-\frac{r^2}{R^2}\right)$$

Granular Soil Pressure

$$p(r) = 2/3\frac{Q}{\pi R^2} \times \left(1+\frac{r^2}{R^2}\right)$$

Cohesive Soil Pressure

FIG. 4.

4.2.2 Assumptions

1) Behaviour of the footing slab is rigid-perfectly plastic.
2) The footing slab is "under-reinforced" (i.e. steel ratio is low), and hence the reinforcement yields prior to the crushing of concrete.
3) Membrane action is neglected.
4) The lever arm of the internal forces is constant.
5) The properties of concrete have no effect on the design.
6) The difference in effective depth of two layers of reinforcement nearer to one face of the slab is neglected.
7) Distribution of soil pressure is known.
8) The footing slab does not fail either due to diagonal tensional or punching shear.

4.2.3 Circular Tapered Footing Slab

Consider a circular footing of radius R, column radius A, isotropically reinforced (i.e. square mesh reinforcement) at bottom and the depth of the footing tapered from the face of the column in such a way that at the centre of the footing both the radial and circumferential moment capacities are $M_o$ (1 $+\alpha$ ) and at edge,

$M_o$, where $\alpha$ is a factor to account for the taper of the footing slab. The general equation of the moment capacity is then,

$$M_{rp} = M_{r\theta} = M_o \; (1 + \alpha \; (1-r/R)) \tag{4.2}$$

where, $M_{rp}$ and $M_{op}$ are radial and circumferential moment capacities respectively.

From the basis of the ultimate load design,

$$M_p = \frac{0.95 \; d \; F_y}{s} \tag{4.3}$$

where, $M_p$ = Moment capacity/unit width

$d$ = effective depth of the footing

$s$ = spacing of the reinforcement.

$F_y$ = yield force/bar.

Here, uniform spacing of the reinforcement is adopted, and hence, the depth of the footing will be directly proportional to the moment capacity. The taper of the footing will be provided in accordance with the varying moment capacity.

Now, there are two failure mode shapes, viz. (i) fan

mechanism and (ii) the partial collapse with an outer rigid region.

$Q$ = Total Load on the footing

$p$ = Uniform Pressure = $Q/\pi R^2$

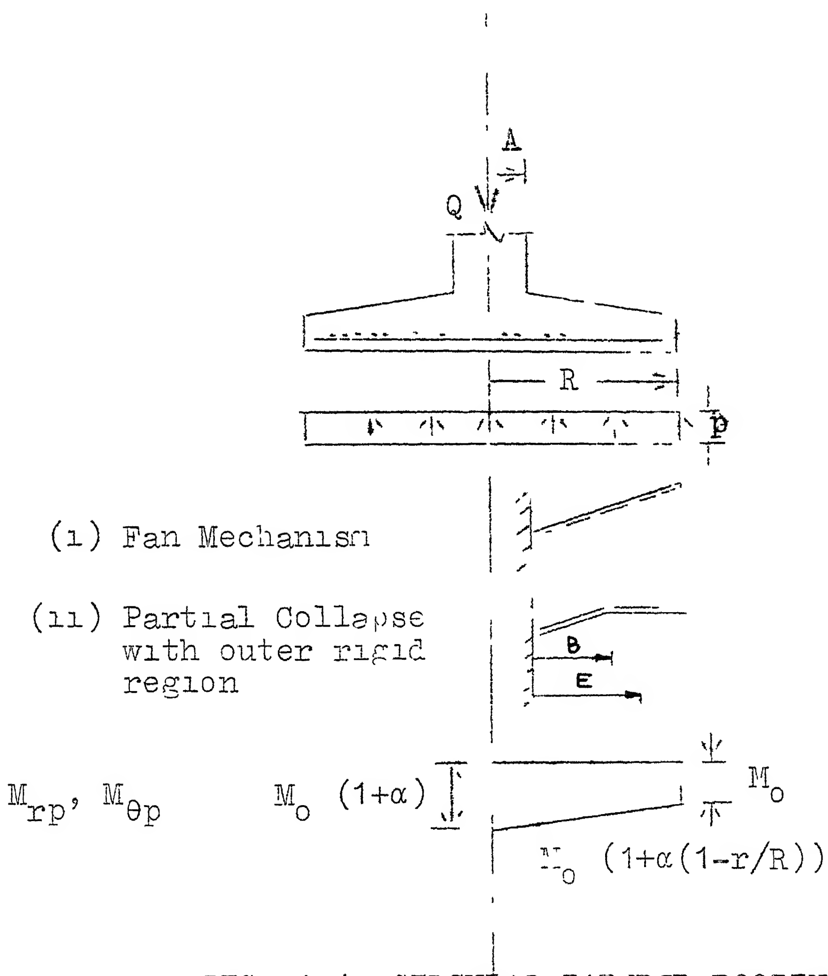



FIG. 4.1 CIRCULAR TAPERED FOOTING

In the figure 4.1,

B = Radius of the circular yield line.

E = Radius of the region boundary at which the value of $M_\theta$ changes

$M_{\theta i}$ & $M_{ri}$ are the circumferential and radial moments respectively in a region i. Introducing the non-dimensional quantities a, b and e = A/R, B/R and E/R respectively and proceeding in the same lines as outlined in article 3.1.1, the moment capacities are determined as follows :

By theorem 3.1,

(i) Fan Mechanism :

The equilibrium equation is

$$(rM_r)'' - M'_\theta = p.r \tag{4.4}$$

Integrating,

$$(rM_r)' - M_\theta = \frac{pr^2}{2} + C_1$$

$$(rM_r)' = M_\theta + \frac{pr^2}{2} + C_1$$

$$= M_o\ (1 + \alpha\ (1 - r/R)) + \frac{pr^2}{2} + C_1 \tag{4.5}$$

$$r\,M_r = M_o\,(1+\alpha\,)_r - \frac{M_o\alpha \quad r^2}{2R} + \frac{pr^3}{6} + C_1 r + C_2 \qquad (4.6)$$

$$M_r = M_o\,(1+\alpha\,) - \frac{M_o\alpha \quad r}{2R} + \frac{pr^2}{6} + C_1 + C_2/r \qquad (4.7)$$

The boundary and continuity conditions are,

$$\begin{aligned} M_r &= M_o\,(1+\alpha\,(1-A/R)) \quad &; \text{ at } r = A \\ M_r &= 0 &; \text{ at } r = R \\ (rM_r)' - M_\theta &= 0 &; \text{ at } r = R \end{aligned} \qquad (4.8)$$

Finally

$$\frac{M_o}{pR^2} = \frac{a^3-3a+2}{3(2+\alpha - \alpha\,a^2)} \qquad (4.9)$$

or $$\frac{M_o\,\pi}{Q} = \frac{a^3-3a+2}{3(2+\alpha - \alpha\,a^2)} \qquad (4.10)$$

(ii) Partial Collapse with Outer Rigid Region :

The region topography is given in fig.(4.1(ii))

$$M_{\theta 1} = \dot{M}_o \ (1 + \alpha \ (1 - r/R)) \quad ; A \leqq r \leqq E$$

$$M_{\theta 2} = 0 \quad ; E \leqq r \leqq R \qquad (4.11)$$

where $E$ is the radius of the region boundary at which the value of $M_\theta$ changes. The boundary and continuity conditions are given by,

$$M_{r1} = M_o \ ( 1 + \alpha \ (1-r/R)) \quad ; \text{ at } r = A$$
$$M_{r1} = 0 \quad ; \text{ at } r = B$$
$$M'_{r1} = 0 \quad ; \text{ at } r = B$$
$$M_{r1} = M_{r2} \quad ; \text{ at } r = E$$
$$(rM_{r1})' - M_{\theta 1} = (rM_{r2})' - M_{\theta 2} \quad ; \text{ at } r = E \qquad (4.12)$$
$$M_{r2} = 0 \quad ; \text{ at } r = R$$
$$(rM_{r2})' = 0 \quad ; \text{ at } r = R$$

$$M_{r1} = M_o \ ( 1 + \alpha ) - \frac{M_o \alpha r}{2R} + \frac{pr^2}{6} + C_1 + C_2/r \qquad (4.13)$$

$$M_{r2} = \frac{pr^2}{6} + C_3 + \frac{C_4}{r} \qquad (4.14)$$

Finally,

$$\frac{M_o}{pR^2} = \frac{1 - b^2}{2(1 + \alpha \ (1-b))} \qquad (4.15)$$

From,

$$M_{r1} = M_o \ ( 1 + \alpha \ (1 - A/R)) \qquad \text{at } r = A$$

$$\frac{M_o}{pR^2} = \frac{3a-a^3-2b^3}{3(a^2-b^2)} \qquad (4.16)$$

is obtained.

Equating the two expressions of $\frac{M_o}{pR^2}$,

$$\frac{1 - b^2}{2(1 + \alpha \ (1 - b))} = \frac{3a-a^3-2b^3}{3(a^2 - b^2)\alpha} \qquad (4.17)$$

Finally,

$$\alpha \ b^4 - 4 \ ( 1 + \alpha ) \ b^3 + 3 \ \alpha \ ( 1 + a^2) \ b^2 - 2a \ \alpha \ (3-a^2) \ b$$

$$+ 6a(1 + \alpha ) - 3a^2 \alpha - 2a^3 \ ( 1+ \alpha ) = 0 \qquad (4.18)$$

A set of values of b is evaluated for $\alpha = 0,1,2,3,4$ and $a = 0.1, \ 0.15, \ 0.20$. For various column sizes and footing tapers the equation (4.18) is solved for b and the [illegible] drawn.

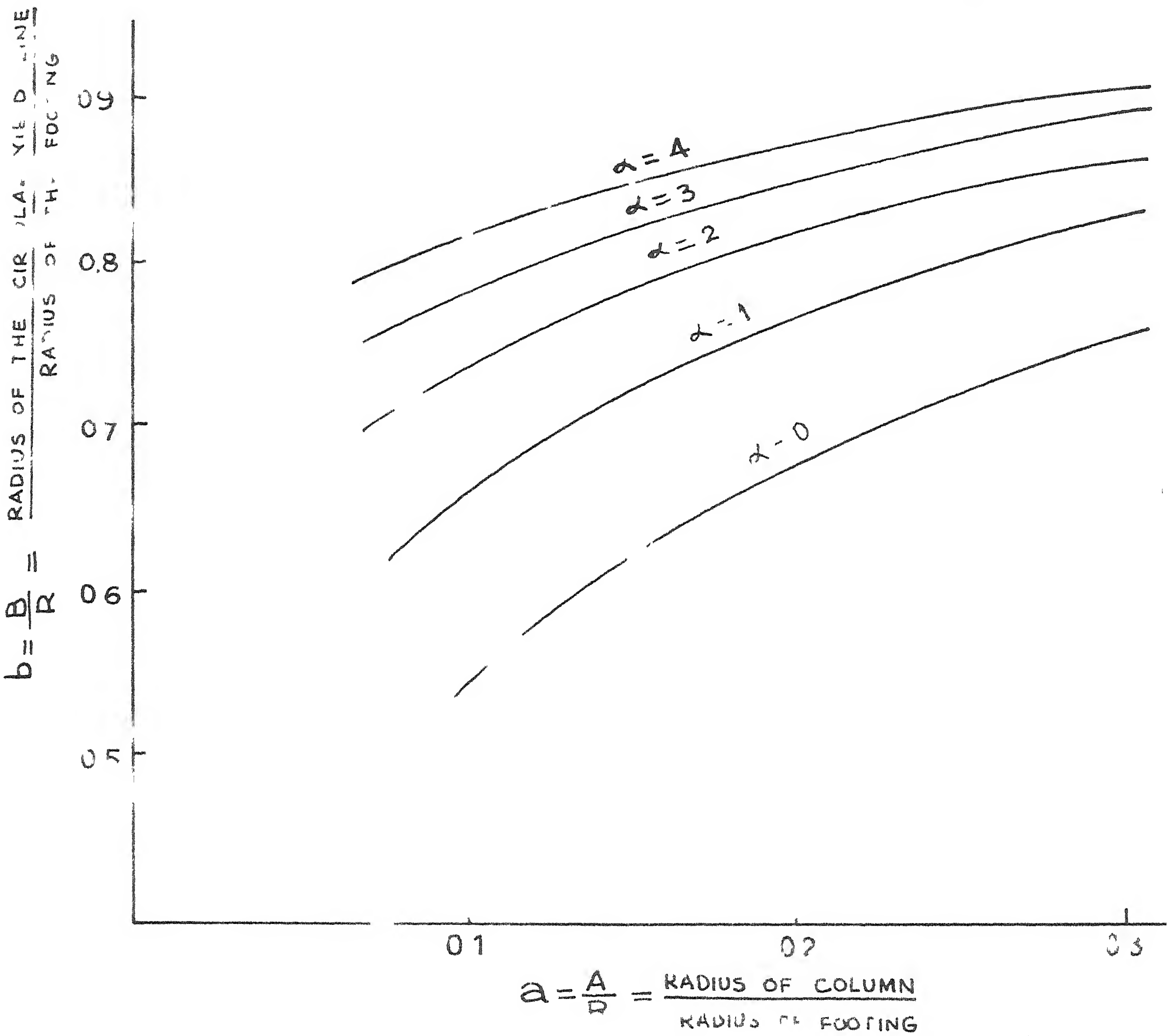



FIG 4A(i) TAPERED FOOTING SLAB

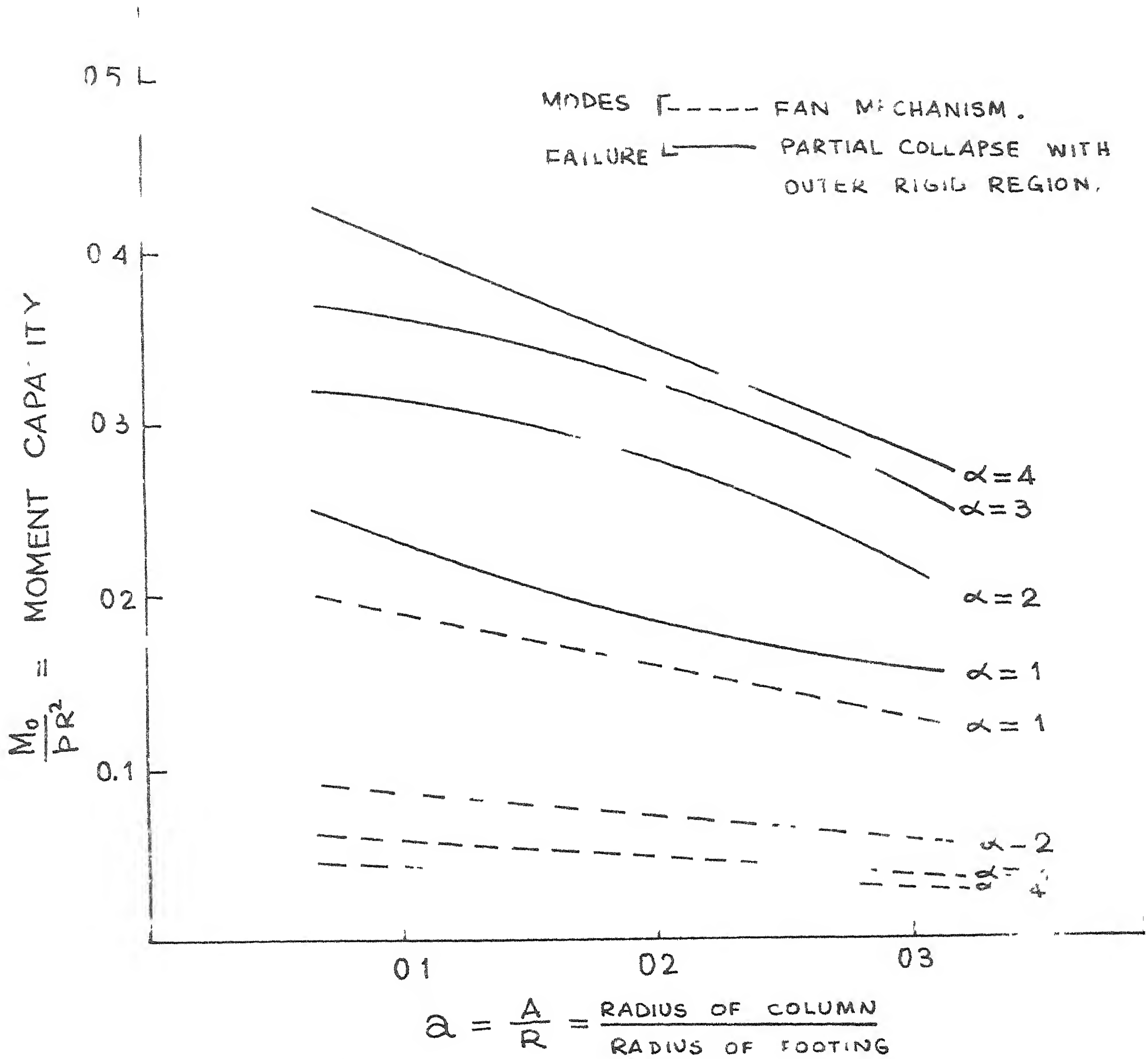



FIG. 4A(ii) TAPERED FOOTING SLAB

# CHAPTER V

## EXPERIMENTAL TESTS ON SIMPLY SUPPORTED REINFORCED CONCRETE CIRCULAR SLABS DESIGNED BY "YIELD EQUALITY METHOD"

### 5.1 INTRODUCTION

This chapter aims to arbite on the "yield equality method". Destructive tests on some five slabs were conducted with four types of reinforcement. First four slabs were designed by the "yield equality method" while the fifth one was reinforced in such a manner as to provide both radial and circumferential moment capacities in accordance with that required by the "elastic analysis". Isotropic and polar network reinforcements were used in the design of the first four slabs.

All the slabs were simply supported along the periphery. The partial uniformly distributed load was applied through a circular steel plate and jack.

An account of the previous tests conducted by others are given in the succeeding article.

## 5.2 PREVIOUS TESTS BY OTHERS

A considerable number of ultimate load tests on isotropic and orthotropic slabs with simple and complex boundary conditions have been performed to justify the use of the yield-line method as a basis for slab design. There have been relatively a few tests on slabs designed by lower bound methods. An elaborate account of the tests by others is given by ADIDAM (1972,2).

Armer (1968,6) examined the behaviour of rectangular reinforced concrete slabs with various support conditions, both at working and ultimate load levels. The slabs tested were designed by a modified version of Hillerborg's strip method.

Muspratt (1969,5) conducted tests on circular footing slabs designed by both upper and lower bound methods and concluded that the radial reinforcement in the slabs was lightly stressed.

ADIDAM (1972,2) conducted tests on some fifteen slabs and arrived at satisfactory results.

## 5.3 AIMS OF EXPERIMENTAL PROGRAMME

The main aim of the tests performed was to provide

further information on the actual behaviour of slabs designed by the "yield equality method". The detailed objectives were to :

1. Determine ultimate loads and collapse modes, and compare these with theoretical predictions.

2. Determine the stress diagram of the steel reinforcement to assess the effects of reinforcement pattern on yielding behaviour.

3. Compare deflections and ultimate load of different designs.

4. Record strain variation at various points with different stages of loading and to show the strain flow in plastic range.

5. Verify the validity of the "yield equality method".

Full report on the experimental tests conducted on the five circular simply supported R.C. Slabs is presented in the following articles.

## 5.4 EXPERIMENTAL TEST REPORT

### 5.4.1 Experimental Apparatus

(5.4.1.1 Loading Arrangement & Reaction Frame

A section through the testing apparatus is shown in Fig. 5A.1 and a general view at the beginning of a test in Fig. (PLATE-II).

The slab was subjected to a partial uniformly distributed load around the axis of symmetry by a 40 cm diameter 'rigid' steel plate. A simply supported condition at the edge of the slab was achieved by placing the slab on a circular ring of steel rod of 22 mm diameter which in turn was supported on a circular 20 cm. thick brick wall 35 cm high. The circular wall was constructed on the structural floor of the laboratory. The circular steel plate was loaded with the help of a Jack which was clamped to the reaction girder. Ultimately the load is transferred through the cross beams and the four Steel Rods to the floor (see Fig. 5.A.1).

5.4.1.2 Measuring Equipment

The pressure in the Jack was recorded from the gauge of the pumping unit. The load applied was determined from the calibration chart of the Jack prepared with the help of a proving ring.

The vertical deflections of the slab were measured by dial gauges 60 mm travel with 0.01 mm divisions; actual slab deflections were obtained by subtraction. Dial gauges were fitted along one diameter only to verify whether the conditions were axisymmetric.

Seven Electrical resistance strain gauges of gauge length 10 mm and gauge factor 2 were used to measure actual steel strains at various stages of loading. A set of indicating and balancing units read out the MICRO STRAINS directly.

Photographs of crack patterns were taken after hanging the slab through the derrick crane.

5.4.1.3 Materials

(A) AGGREGATES

The maximum aggregate size used in the concrete was 12 mm.

(B) CEMENT

Portland Cement Type A was used.

(C) STEEL REINFORCEMENT

The reinforcement used for all slabs was 6.6 mm.

## PHYSICAL PROPERTIES OF AGGREGATES

### TABLE 1

| PROPERTY | FINE AGG. | COARSE AGG. |
|---|---|---|
| (i) Bulk density | 1.673 Kg./Lit. | 1.57 Kg./Lit. |
| (ii) % Absorption | 2-3% | 1% |
| (iii) % Passing through | | |
| 18 mm | | 100 |
| 12 mm | | 70.4 |
| 9 mm | | 32.2 |
| 6 mm | | 6.4 |
| 4.5 mm | | 2.1 |
| IS SIEVE : | | |
| 480 | 99.32 | |
| 240 | 91.60 | |
| 120 | 70.38 | |
| 60 | 48.74 | |
| 30 | 6.16 | |
| 15 | 2.50 | |
| Pan | | |

diameter mild steel. This had an average yield force of 1125 Kgs. The free elongation diagram for these bars generally showed a flat yield plateau after first yield. A stress-strain diagram obtained from the testing of a steel rod with the help of strain gauges and later verified by the autographic plotted idealizing the area to remain same is shown in the figure (5.A.2.a). Bottom steel mesh was placed on 1 5 cm. bars to provide 1.5 cm cover (Fig. 5 3 (i)).

(D) MIX DETAILS

Three trial mixes were designed according to the ACI Standard Mix Design Procedure for an average target strength of 250 $Kg/cm^2$ at 7 days. (see table 2).

5.4.2 Basis Of Design

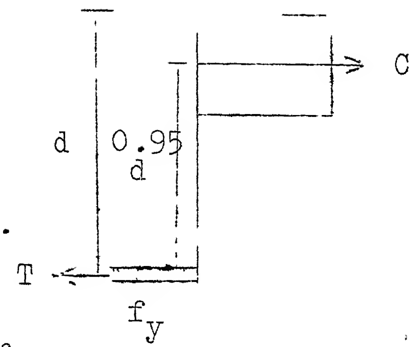
In accordance with the lower bound theorem, the steel in all slabs was proportioned to be capable of sustaining an equilibrium distribution of moments. All slabs were 7 cm thick. The lever arm of the internal form for a set of bars with the same cover was assumed constant, even though local concentrations of steel strictly requires a reduction. Because of the range of steel ratios, a lever arm of 0.95 times the effective depth was assumed for design.

TABLE 2

| | MIX I | MIX II | MIX III |
|---|---|---|---|
| Cement | 1 | 1 | 1 |
| Sand | 1.50 | 1.6 | 1.85 |
| Coarse aggregate | 1.72 | 1.8 | 2.00 |
| Water Cement Ratio | 0.40 | 0.45 | 0.50 |
| 7 days compressive cube strength ($Kg/cm^2$) | 225 | 249.3 | 184.6 |
| 7 days compressive cylinder strength ($Kg/cm^2$) | | 189.9 | 161.0 |
| 7 days Bond Stress ($Kg/cm^2$) | | 17.88 | 13.7 |
| Modulus of Rupture | | 22.40 | 18.8 |
| Slump (Expected cm) | | 8 | 8 |
| Slump (Observed cm) | | 9 | 4.5 |
| Young's Modulus of Elasticity $E_c$ | | $0.34 \times 10^6$ $Kg/cm^2$ | |
| Poission's Ratio | | 0.22 | |
| $E_s/E_c$ | | 6 | |

Mix II was found to be useful for the purpose.

The tensile force T is given by




$$T = \frac{A_s\ f_y . b}{s} = F_y . b.$$

where $F_y$ is the yield force/bar.

From equilibrium, C = T

and the moment per unit width of slab is

$$M_p = \frac{T\ 0.95 .\ d}{b} = \frac{0.95\ d .\ F_y}{s}$$

The stresses assumed on a slab cross-section are shown in the figure these refer to a width b containing bars of cross-sectional area $A_s$ at spacing s. The lever arm of the internal forces is assumed as 0.95 times the effective depth d as suggested by Wood.

5.4.2.1 Theoretical Collapse Loads

Consider a circular slab of radius R partly subjected to a uniform pressure p over the region 0 r F (i.e. around the axis of symmetry. For a slab istropically reinforced at the bottom only,

Total Collapse load (TC)

$$p = \frac{6M}{f^2 (3-2f) R^2}$$

Partial Collapse with an Inner Rigid Region (PO)

$$p = \frac{4M}{f^2 R^2}$$

Partial Collapse with an Outer Rigid Region (PI)

$$p = \frac{6Ma}{(3a-2f) f^2 R^2}$$

### 5.4.2.2 Example of Slab Design : Description of Slabs Tested

Normal code requirements, such as span to depth ratios, minimum and maximum spacing and percentage of reinforcing steel, and minimum cover, were not strictly adhered to. The primary purpose of the slab experiments was to compare the behaviour of slabs having different patterns of reinforcements, and other considerations were of secondary importance.

SLABS - $S_2$, $S_4$

Consider Slab $S_2$, assumed to be isotropically reinforced with equally spaced bars in two directions at right angles. The bars closest to the tension face have an effective depth of (7.0 - 1.5 - 0.33=) 5.17 cm and the other bars an effective depth of (5.17-0.66 = ) 4.51 cm.

$P = \pi F^2$ $p = 5$ t $\quad F_y = 1125$ Kgs.

$R = 100$ cms. $\quad d = 5.17$

$F = 20$ cms

$$M_p = \frac{pF^2 \ (3R-2F)}{6R} = 690 \text{ Kg. cm.}$$

$$M_p = \frac{0.95 \ d \ F_y}{s}$$

$\therefore$ s = 8 cms.

Provide 6.6 mm bar at 8 cm c/c both way as shown in Fig. 5A.3 (b).

In Slab $S_4$

10 cm c/c bothway square mesh (isotropic) reinforcement Fig. (5A.3(d)) was provided just to show that they behave in a similar way so far as the deformations, ultimate load and collapse mode are concerned.

SLAB - $S_3$ was reinforced with the polar net of reinforcement shown in Fig. (5A.3(c)). The circumferential steel was placed to provide uniform $M_{p\theta}$ throughout the slab. Radial bars were curtailed in accordance with the $M_r$ diagram.

SLAB - $S_5$ was reinforced (approximately) to meet the moment capacity requirements of elastic design. Fig. 5A.3.(e)

5.4.2.3 Casting of Slabs

The slabs were cast on wooden form work with G.I. Sheet lining, compacted by needle vibrator and screened with a wooden plank to correct thickness. The form was removed after a moist sand bag curing of 7 days and later the curing continued till a day before the test. The strain gauges were fixed on steel reinforcement before tying the bars.

### 5.4.3 Testing Procedure

The slab was loaded in increments of 500 psi pressure ( 500 Kgs ). The deflections were measured when the movement had stopped.

The final crack patterns were photographed after marking the cracks with black ink.

Cubes were tested on the same day as the slab.

The deflections and the strains (on the steel reinforcement) were recorded at every 500 psi pressure.

### 5.4.4 Experimental Results

The details of the experimental results are presented in a tabular form as shown in Table 3.

SLAB $S_1$ failed due to punched shear suddenly at 3.75 tons before any observations of dial gauges could be made.

SLAB $S_2$ behaved very well upto the design ultimate load which can be seen in fig. 5.9 (Plate IX) the steel strains flow with loads prior to failure. The failure was almost sudden at 8.3 tons (including the 600 Kgs.

TABLE 3 TEST RESULTS

| SLAB | Age at test, days | Cube Strength of Concrete at test age Kg./cm$^2$ | | Steel Yield for bars 6.6 mmØ Kg./cm$^2$ | Design Ultimate Load, tons | Actual Ultimate Load tons | Maximum Deflection (Elastic Range) mms | Mode of failure |
|---|---|---|---|---|---|---|---|---|
| | | Field Cured | Water Cured | | | | | |
| $S_1$ | 30 | 415 | 415 | 2220 | 3.100 | 3.750 | 4.0 | Punched Shear |
| $S_2$ | 19 | 341 | 359 | 2220 | 5.000 | 8.300 | 4.0 | Total Collapse |
| $S_3$ | 17 | 325 | 325 | 2220 | 5.000 | 5.300 | 2.4 | Partial Collapse Inner Rigid Region |
| $S_4$ | 11 | 296 | 263 | 2220 | 5.000 | 8.200 | 4.2 | Total Collapse |
| $S_5$ | 9 | 316 | 313 | 2220 | 5.000 | 5.300 | 2.3 | Partial Collapse Inner Rigid Region |

weight of slab). The load-deflection curve in fig. 5A.4 (p.5 (iv)) shows the behaviour for a number of points on the slab. The deflection and steel stress profiles are shown in fig. 5A.8 & 5A.9 (p. 5 (viii)) respectively.

SLABS $S_2$ and $S_4$ were identical in all respects and their behaviour was more or less the same. The photographs of the crack patterns are shown in fig. 5.4 and 5.5. Total collapse occurred well after reaching the design ultimate load.

SLAB $S_3$ collapsed partially with inner rigid region well after the design ultimate load. The failure was almost sudden.

SLAB $S_5$ also like slab $S_3$ collapsed partially with an inner rigid region after the design ultimate load was reached. The failure was almost sudden.

5.5 GENERAL DISCUSSION OF TEST RESULTS AND CONCLUSIONS :

Comparing the load-deflection curves of the slabs $S_2$ and $S_4$ it is found that the slabs show identical behaviour in the elastic range but in the plastic range the deflections increase at lesser rate in the case of Slab $S_2$ - the slab with greater amount of reinforcement.

(see Fig. 5A.8). Fig. 5A.9 shows the similar steel stress profiles of the two slabs, the only difference being that in Slab $S_4$ ( the slab with lesser amount of reinforcement) the steel stresses are a little bit higher. The failure pattern of both the slabs corresponds to the predicted one, that is, the total collapse.

The Slabs $S_3$ and $S_5$ behaved in a similar way. The load - deflection curves (Fig. (5A.5) and (5A.7)) are almost identical in the elastic range and Slab $S_3$ in the plastic range show marked unrestricted plastic flow. The ultimate loads are same for both the slabs.

From these few tests very useful conclusions can be drawn. As the design of slabs was based on the "Yield Equality Method", this method was shown to be reliable and valid. This can be stated from the fact that the predicted and observed collapse loads and modes are well in agreement.

The Slab $S_5$ reinforced as per the requirements by the elastic theory had the same collapse load as that of the Slab $S_3$ designed by "Yield Equality Method". The total length of reinforcement in Slab $S_5$ was 5,231 cms. while in Slab $S_3$ it was 4,217 cms. This shows almost 20%

saving in the steel when designed by the "Yield Equality Method" Even the isotropically reinforced slabs compared per unit collapse load show about 15% reduction in steel.

The Slabs $S_1$, $S_3$ and $S_5$ failed in the post - ultimate stage as a result of punching shear. This is a secondary phenomenon - a usual type of failure for slabs subjected to concentrated loads. Yielding of reinforcement, once starts, produces wide cracks in the tension zone making the slab loose its shear resistance and finally the slab fails through punching shear.

## 5.6 SUGGESTIONS FOR FUTURE RESEARCH

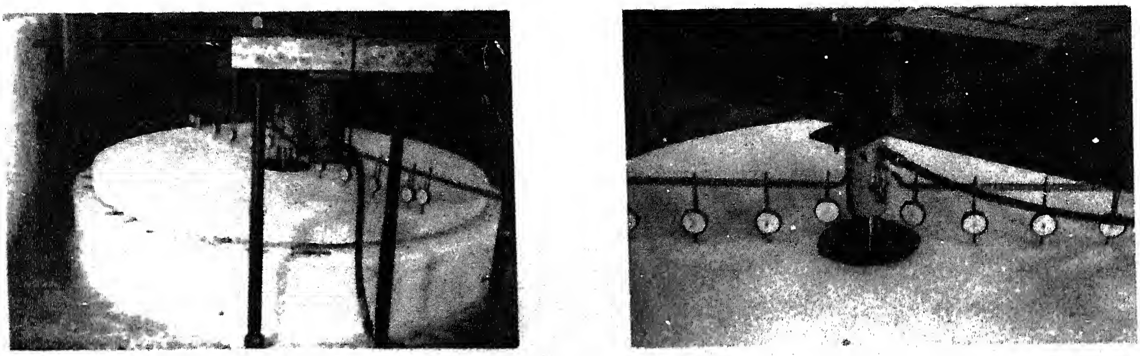
1. The non-axisymmetric slabs could be analysed by modifying the "Yield Equality Method", considering the most general equilibrium equation. Various non-axisymmetric boundary and loading conditions need be tried in the light of modified "Yield Equality Method".

2. Tapered axisymmetric reinforced concrete footing slabs may be analysed for non-uniform foundation pressures such as for granular, cohesive soils, etc.

3. Further tests on circular slabs with various other loading conditions viz. uniformly distributed load circular line load, could be conducted to prove the versatility of the "Yield Equality Method".

(a) (b)

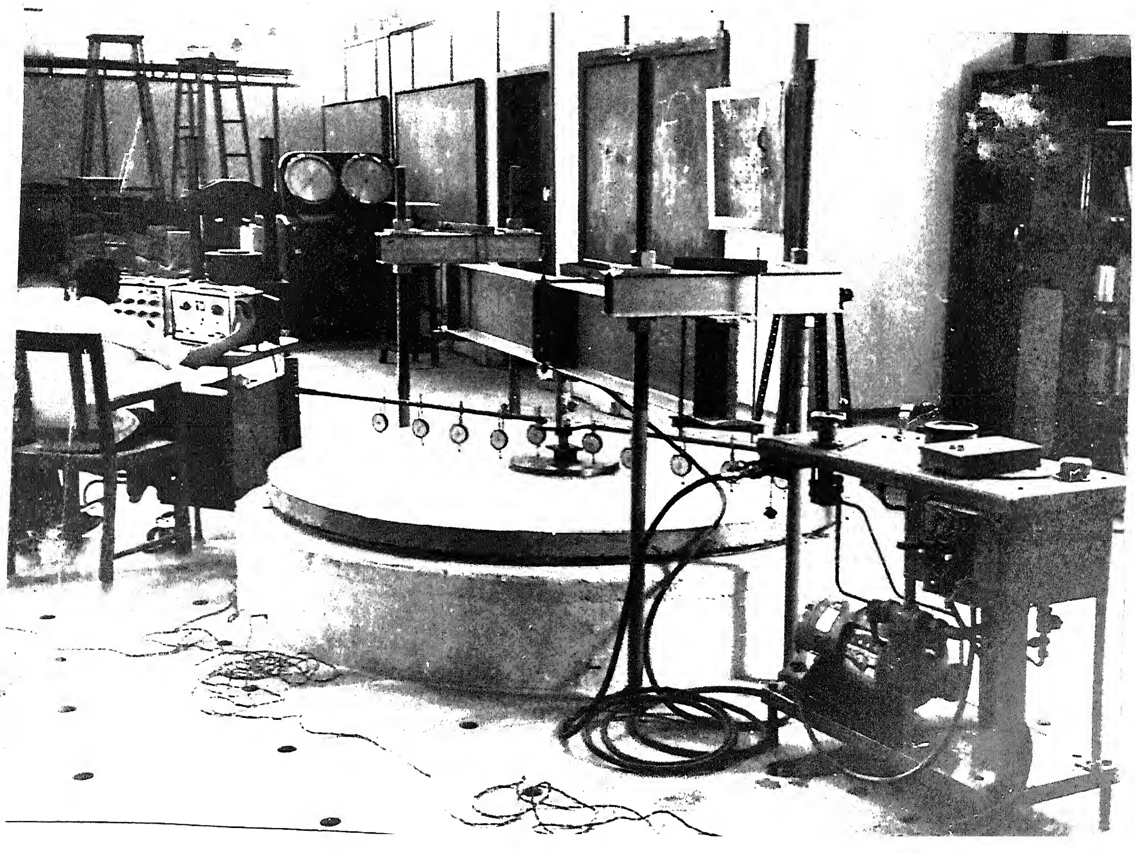
FIG. 5·1 EXPERIMENTAL SET-UP FOR TESTING OF SLABS

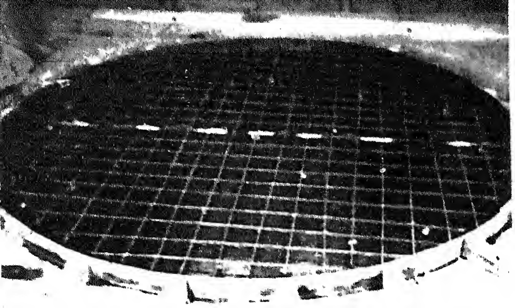


(a) (b)

FIG. 5·2 PUNCHING SHEAR FAILURE OF SLAB $S_1$

PLATE-





(a)

(b)

FIG. 5·3(i) REINFORCEMENTS IN THE MOULD WITH E.R.S. GAUGES

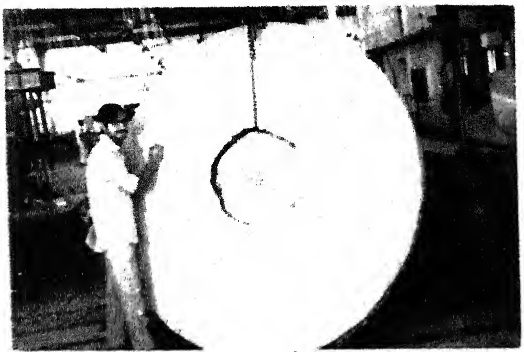


FIG. 5·3 (ii)

THE AUTHOR WITH SLAB - $S_5$

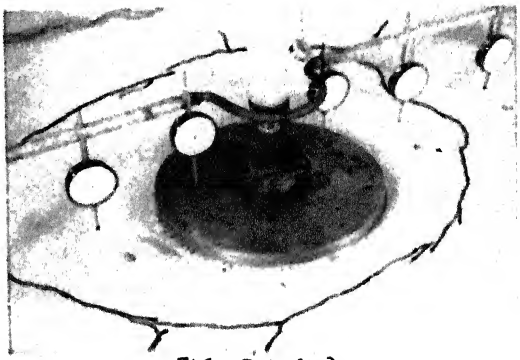
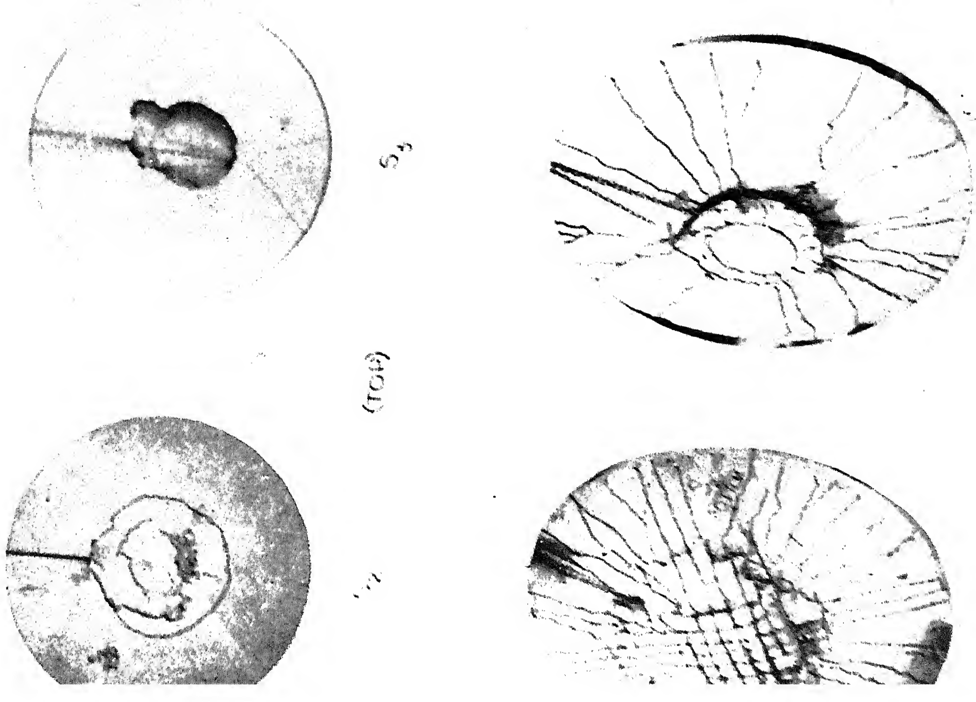


FIG. 5·3 (iii)

SLAB - $S_4$ AFTER FAILURE

PLATE - IV

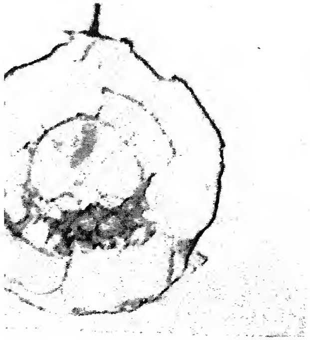


$S_3$

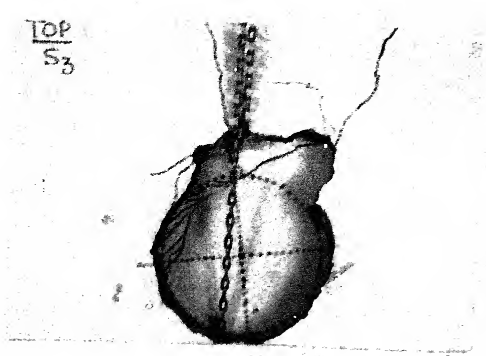
(TOP)

(BOTTOM)

PLATE - VI



$S_2$




$S_3$

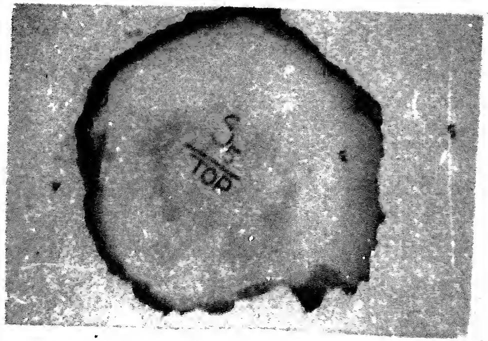
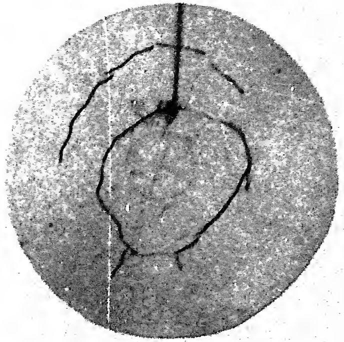


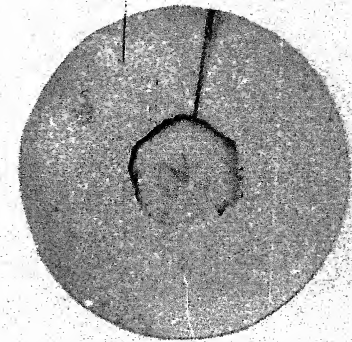
FIG. 5.6 TYPICAL FAILURE DETAILS

PLAT



$S_4$

(TOP)



$S_5$

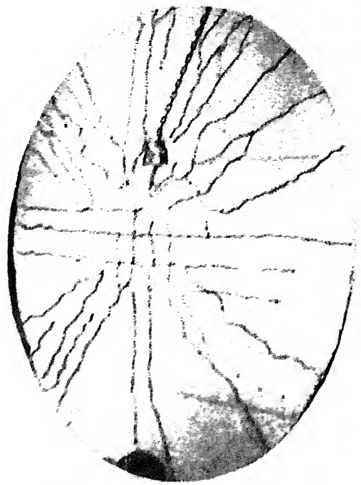
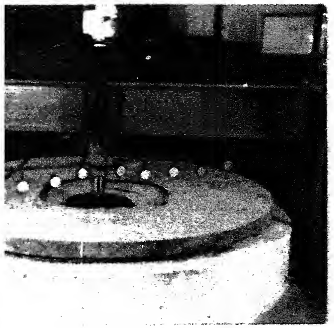
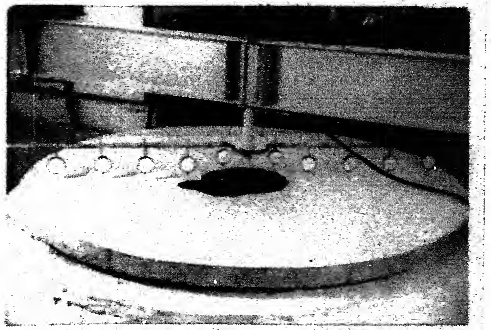


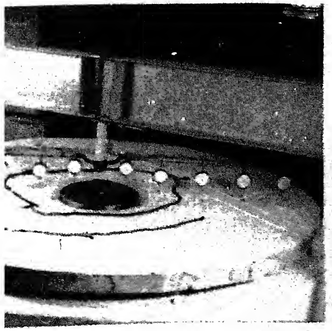
PLATE - VIII



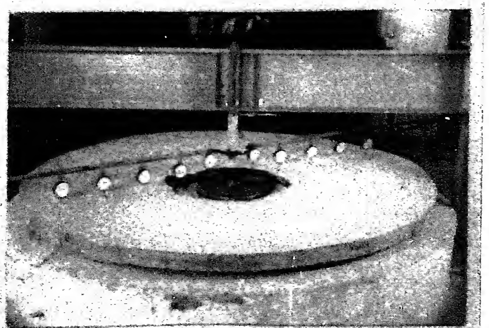
$S_2$



$S_3$



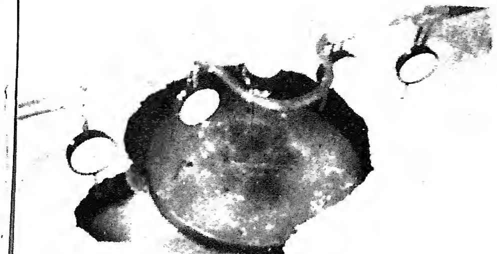
$S_4$



$S_5$

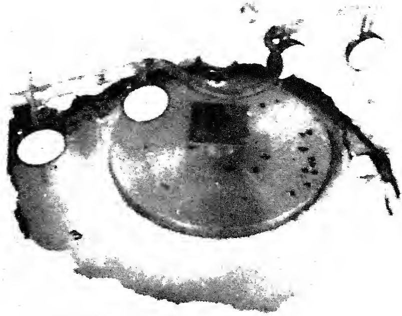
FIG. 5.8 TYPICAL FAILURE DETAILS

PLATE -

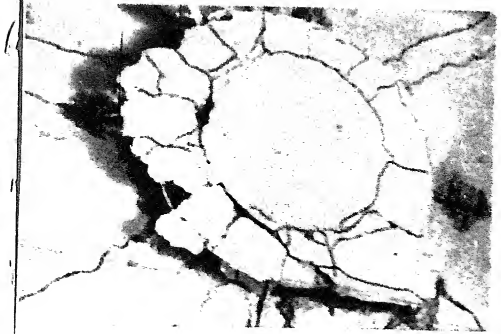


$S_3$

(TOP)



$S_5$



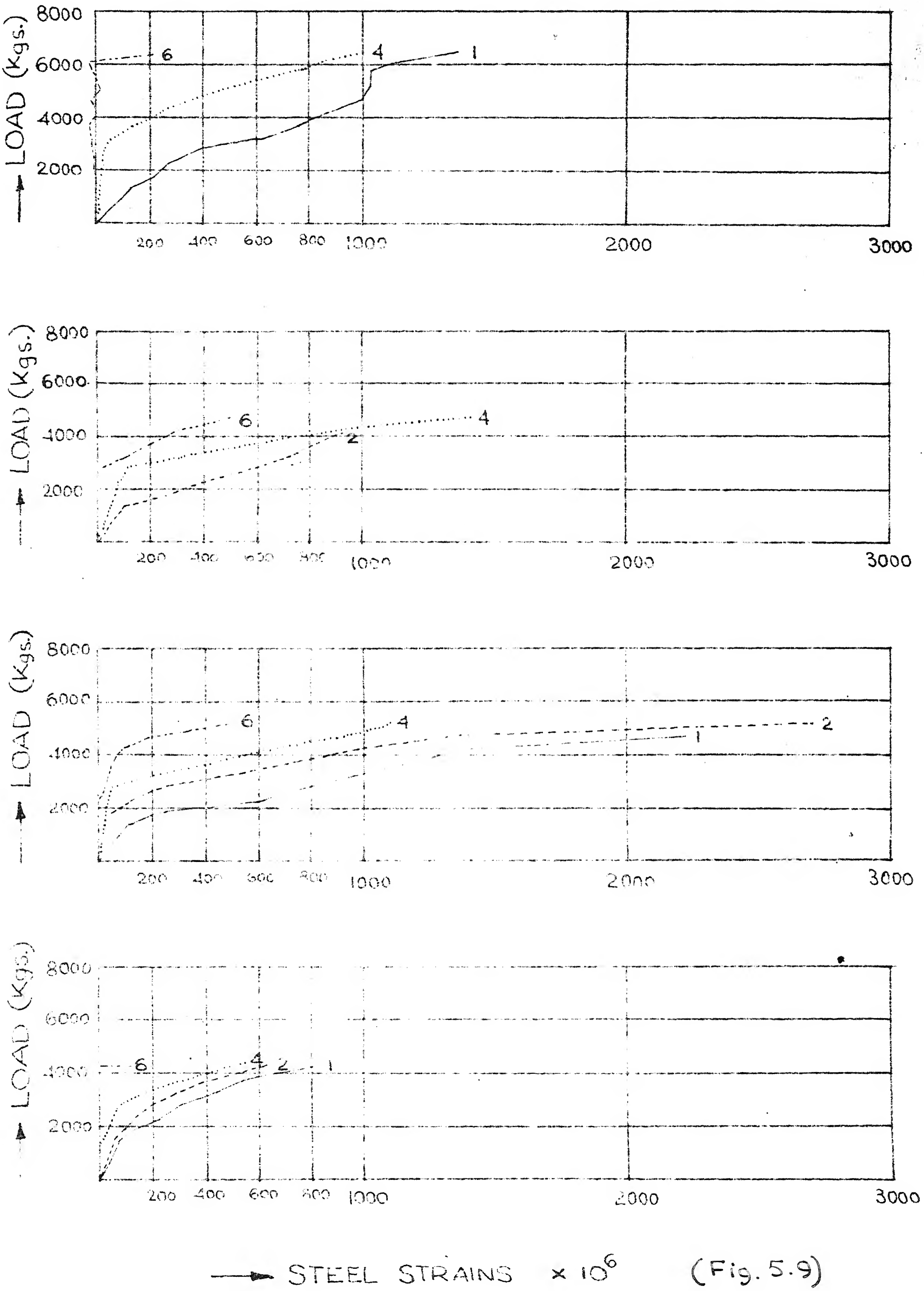
$S_3$

(BOTTOM)


$S_5$

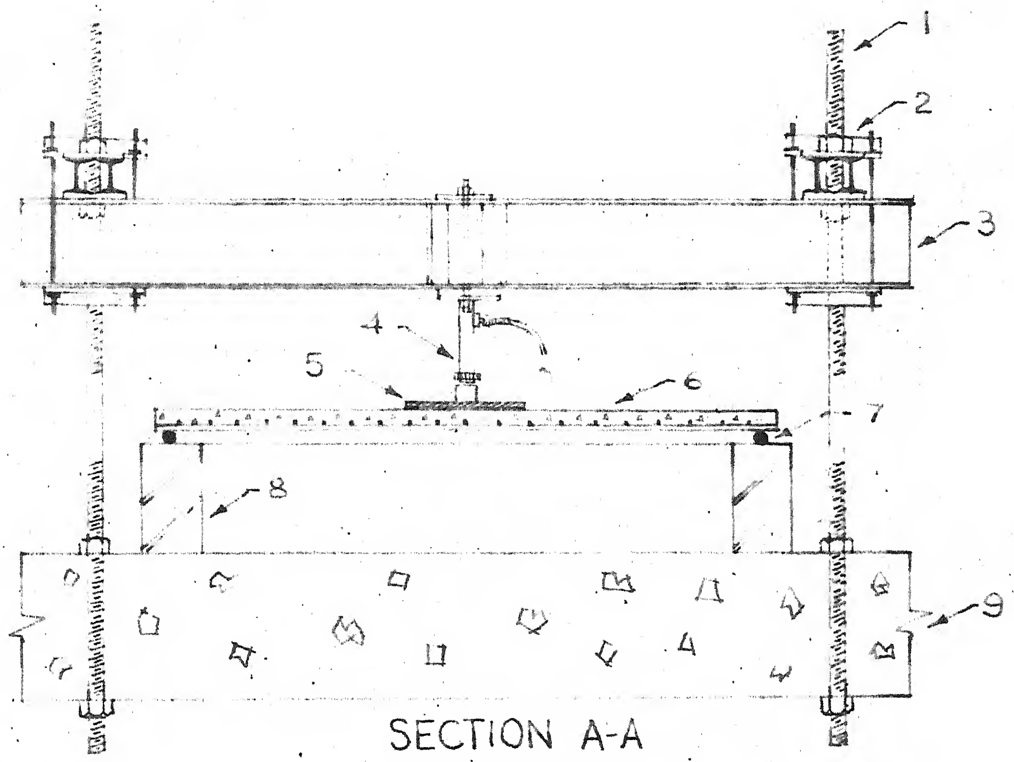
FIG. 5.7 TYPICAL FAILURE DETAILS

PLATE

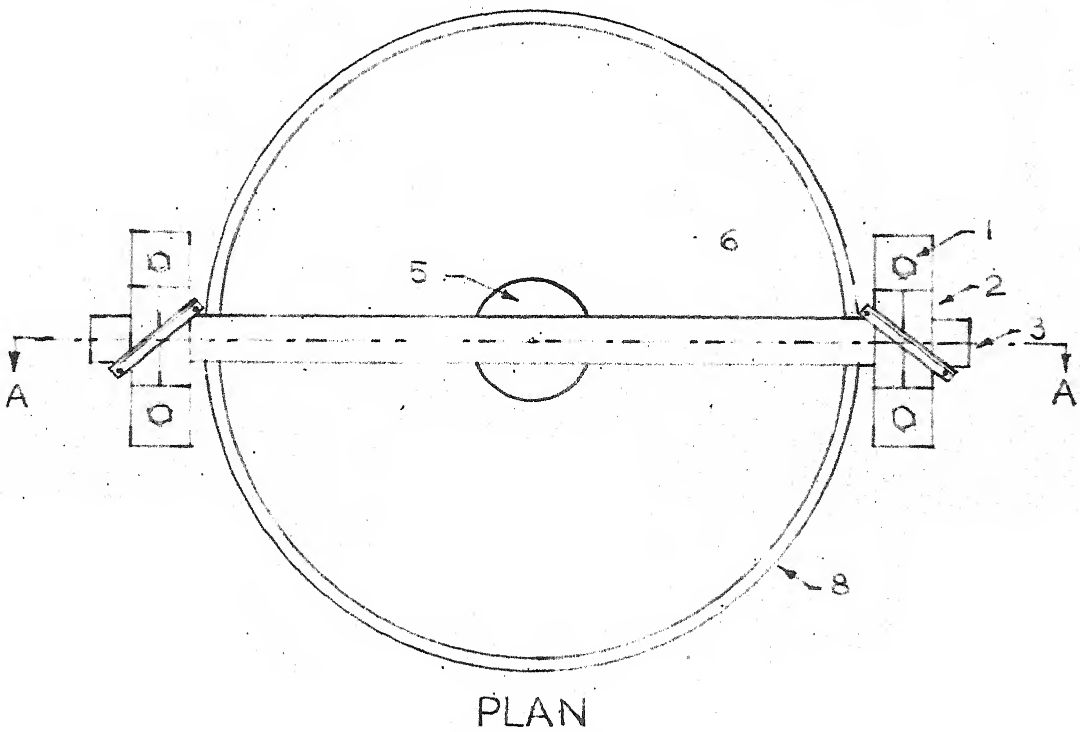


LOAD (Kgs.)
8000
6000
4000
2000
200
400
600
800
1000
2000
3000
6
4
1
LOAD (Kgs.)
8000
6000
4000
2000
200
400
600
800
1000
2000
3000
6
4
2
LOAD (Kgs.)
8000
6000
4000
2000
200
400
600
800
1000
2000
3000
6
4
2
1
LOAD (Kgs.)
8000
6000
4000
2000
200
400
600
800
1000
2000
3000
6
4
2
1
STEEL STRAINS × 10^6

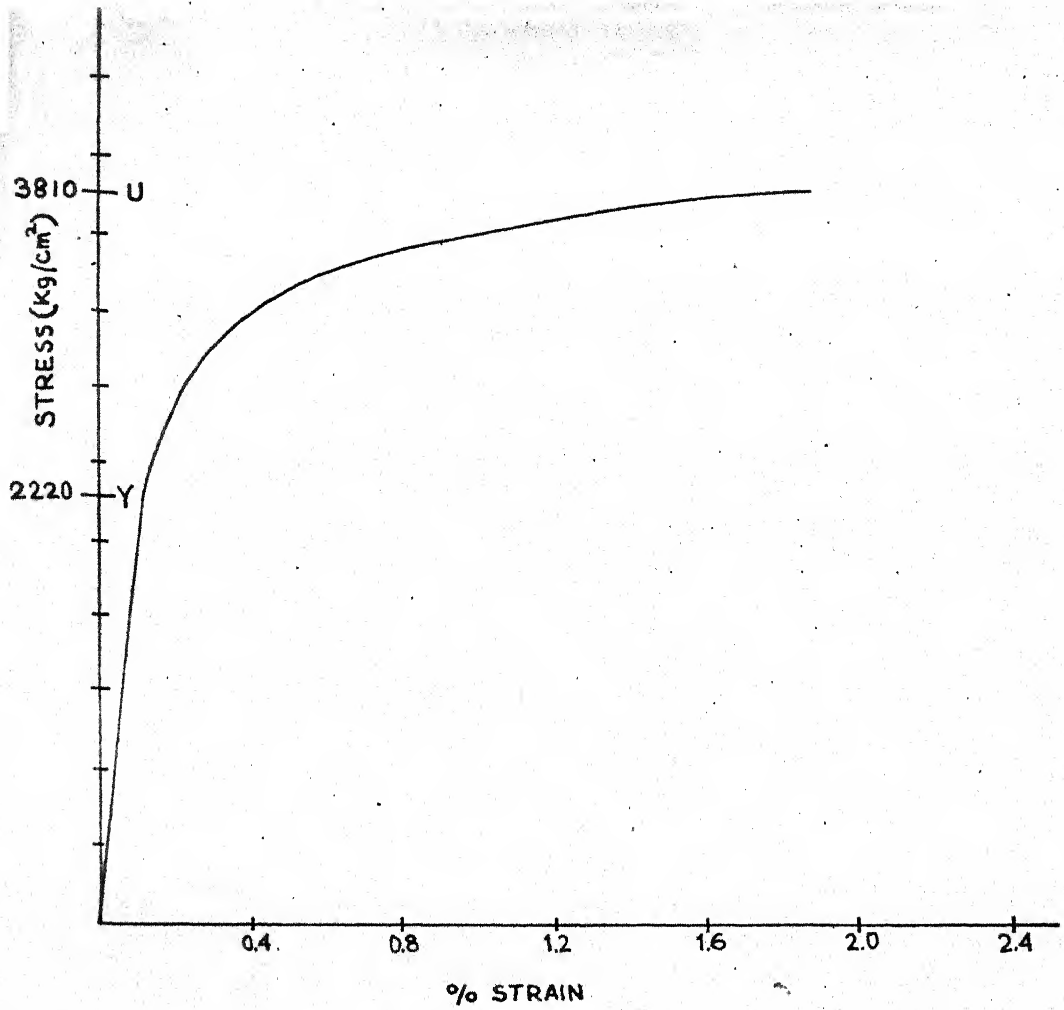

(Fig. 5.9)



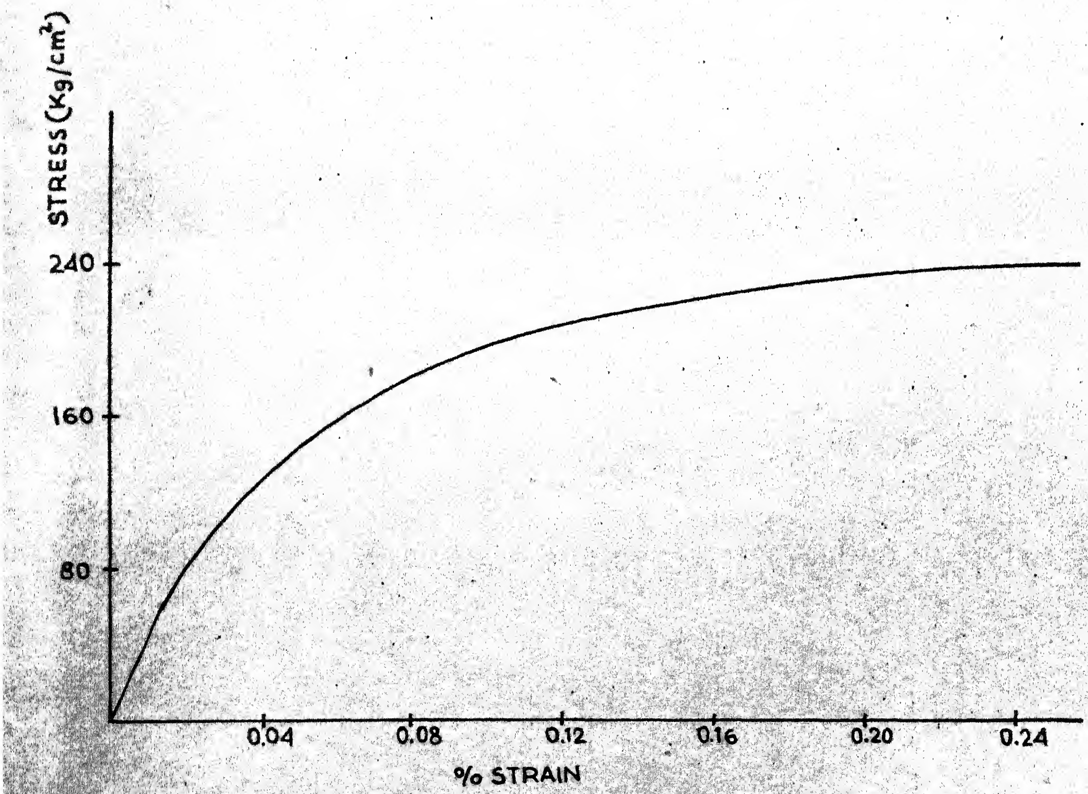

. 40 mm ϕ STEEL ROD
. CROSS GIRDER 2 NOS IS 120
. REACTION GIRDER IS 300
. JACK (CAPACITY 10 TONS)
5. 40 cm DIAMETER 22 mm THICK STEEL PLATE OVER PAD
6. TEST SLAB (2·1 M DIAMETER AND 7 cm THICK)
7. 2 M c/c DIAMETER STEEL RING OF 22 mm ϕ ROD
8. CIRCULAR SUPPORT WALL 35 cm HIGH OF BRICK OVER TEST FLOOR AND LEVELLED AT TOP BY CONCRETING (20 cm WIDTH)
9. LABORATORY TEST FLOOR




IG. 5A·1. Experimental Lay-out For Testing Partially Loaded And Simply Supported Circular Slab.

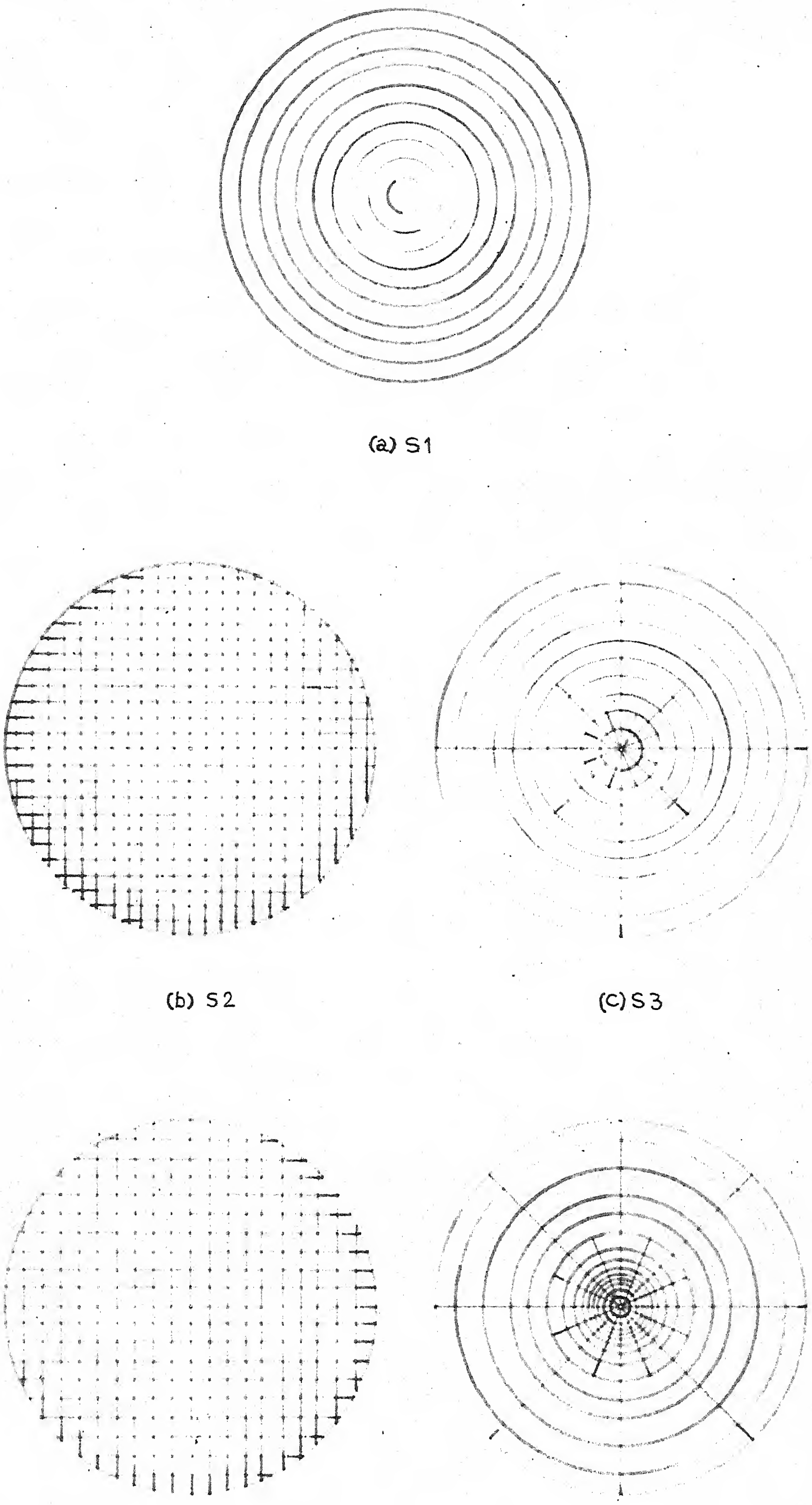



(a) Steel Reinforcement




(b) CONCRETE (MIX 1:1·6:1·8 W.C. RATIO 0.45)

Fig. 5A.2 STRESS - STRAIN DIAGRAMS



(a) S1

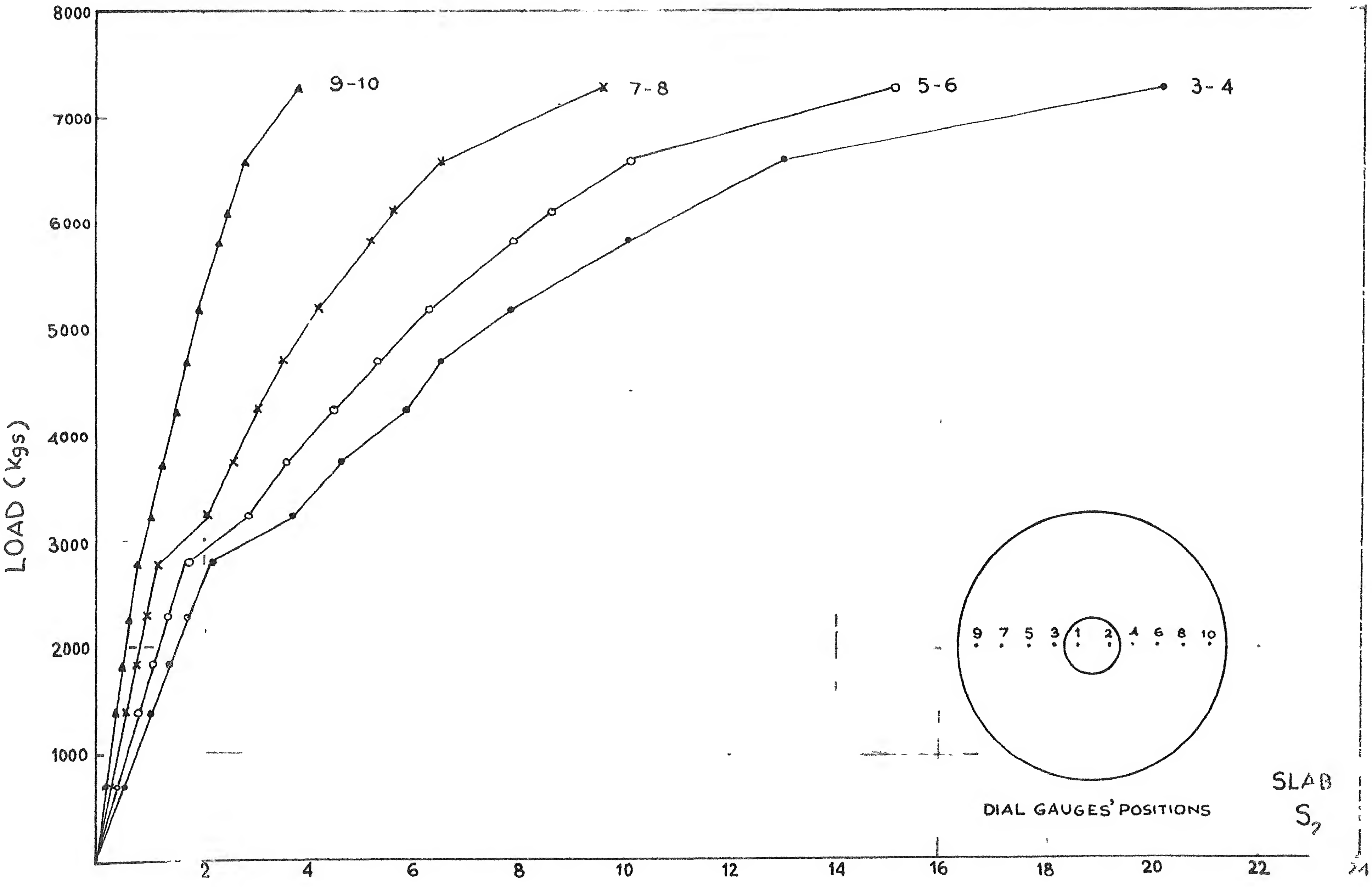
(b) S2

(c) S3

(d) S4

(e) S5

Fig.5A.3 Reinforcement Patterns for Circular Slabs


LOAD (Kgs)
8000
7000
6000
5000
4000
3000
2000
1000
9-10
7-8
5-6
3-4
9 7 5 3 1 2 4 6 8 10
DIAL GAUGES' POSITIONS
SLAB
$S_2$
2
4
6
8
10
12
14
16
18
20
22
24
DEFLECTION (mms)
Fig 5A.4

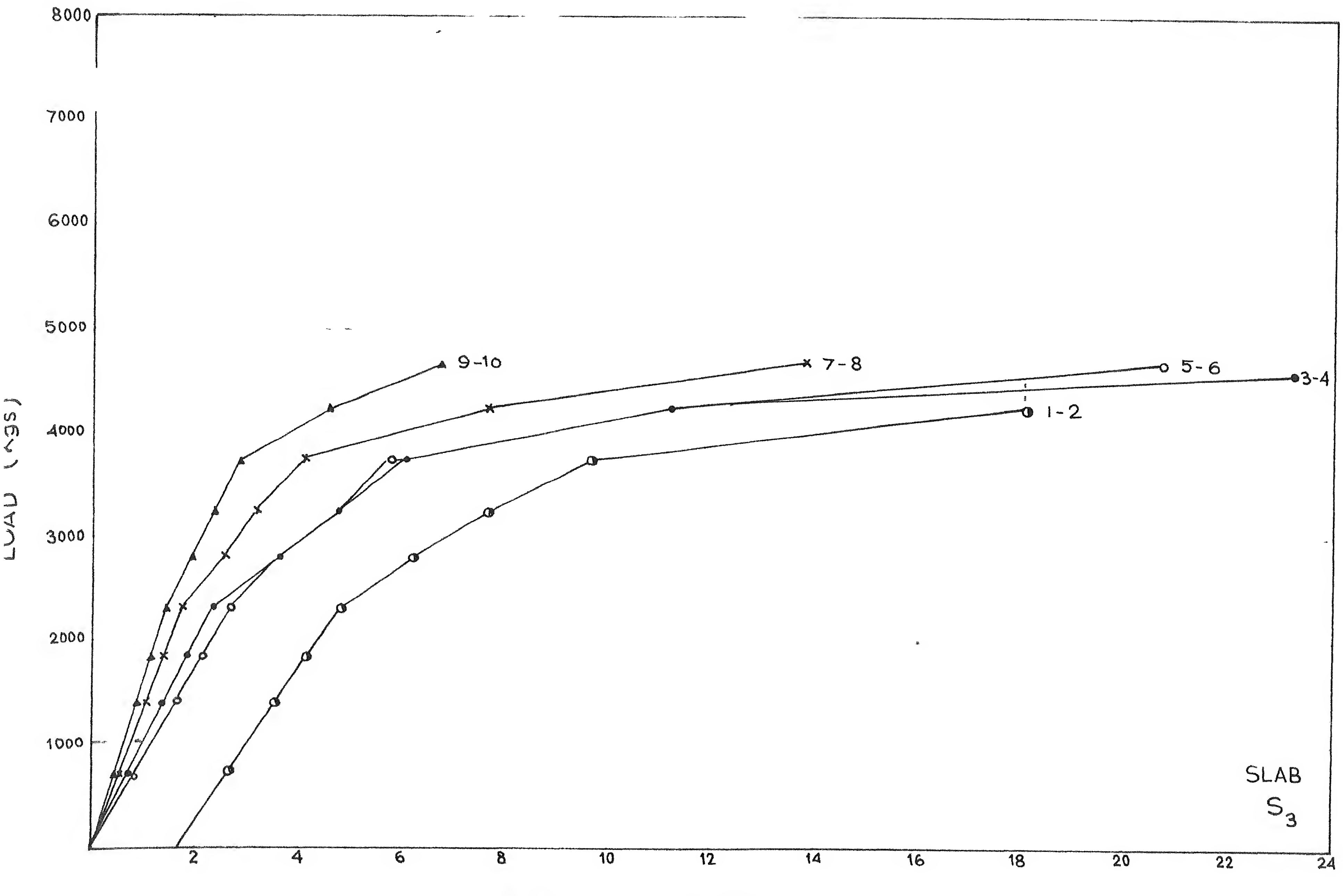

8000
7000
6000
5000
4000
3000
2000
1000
LOAD (KGS)
9-10
7-8
5-6
3-4
1-2
SLAB
$S_3$
2
4
6
8
10
12
14
16
18
20
22
24
DEFLECTION (mms.)

Fig 5A 5

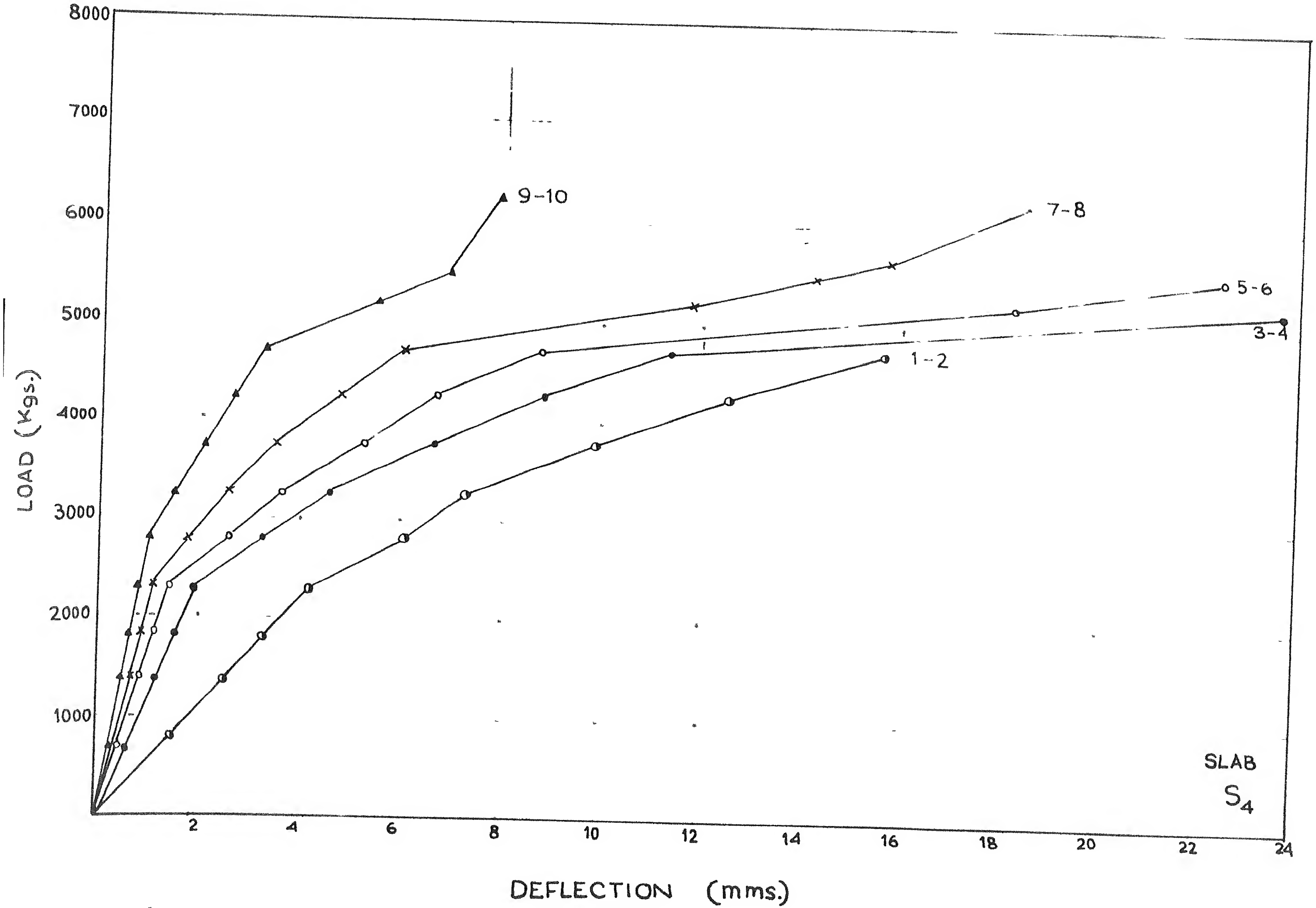
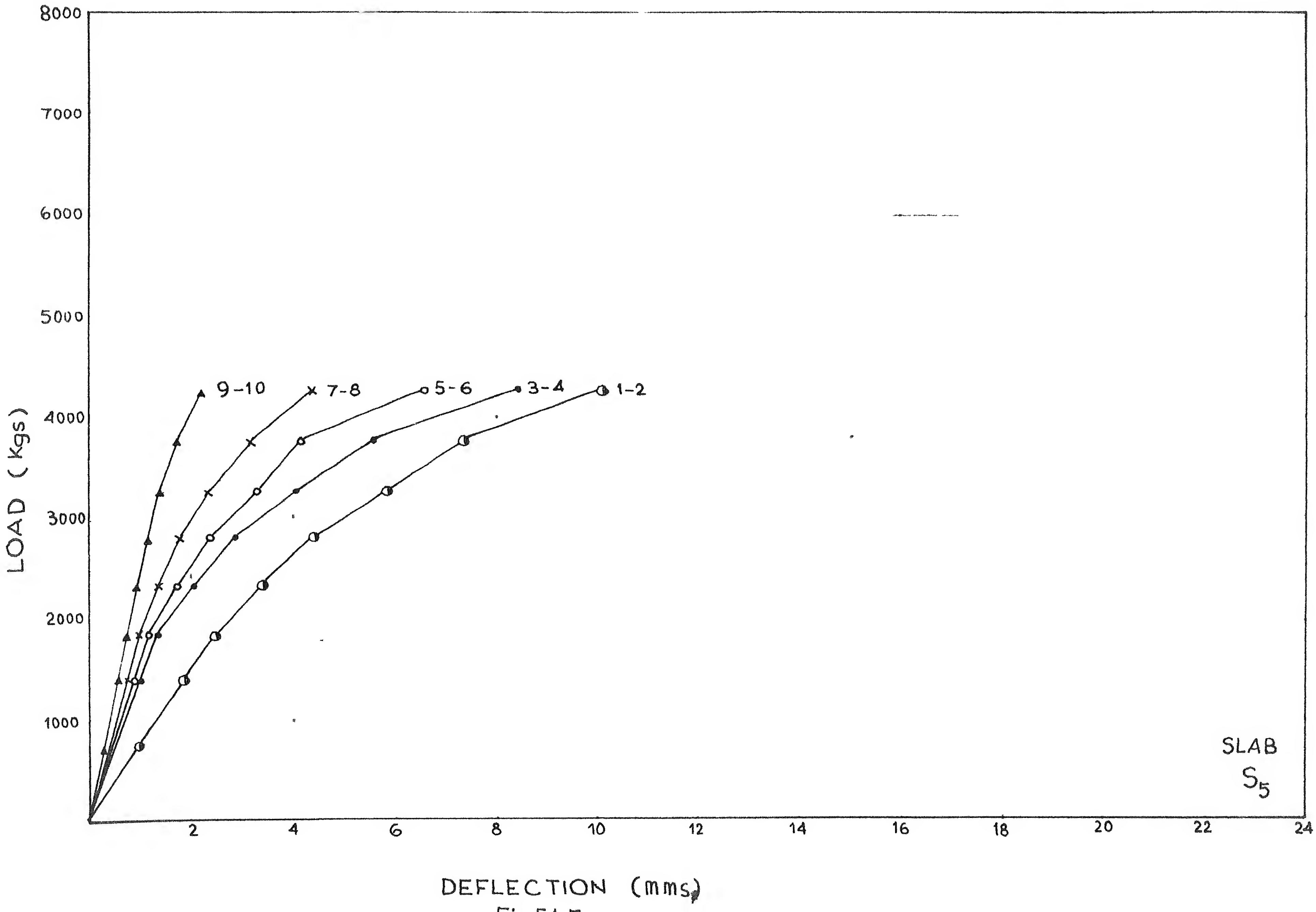
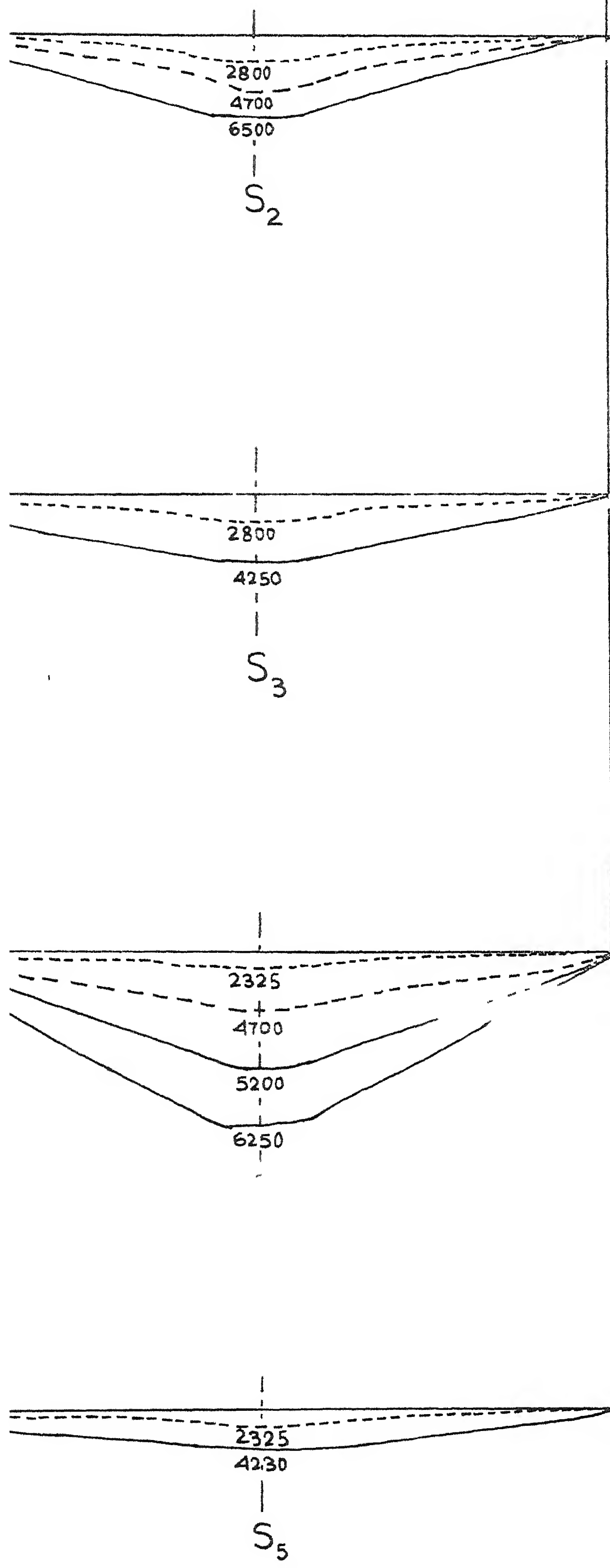

8000
7000
6000
5000
4000
3000
2000
1000
LOAD (Kgs.)
2
4
6
8
10
12
14
16
18
20
22
24
DEFLECTION (mms.)
9-10
7-8
5-6
3-4
1-2
SLAB
$S_4$

Fig 5A 6

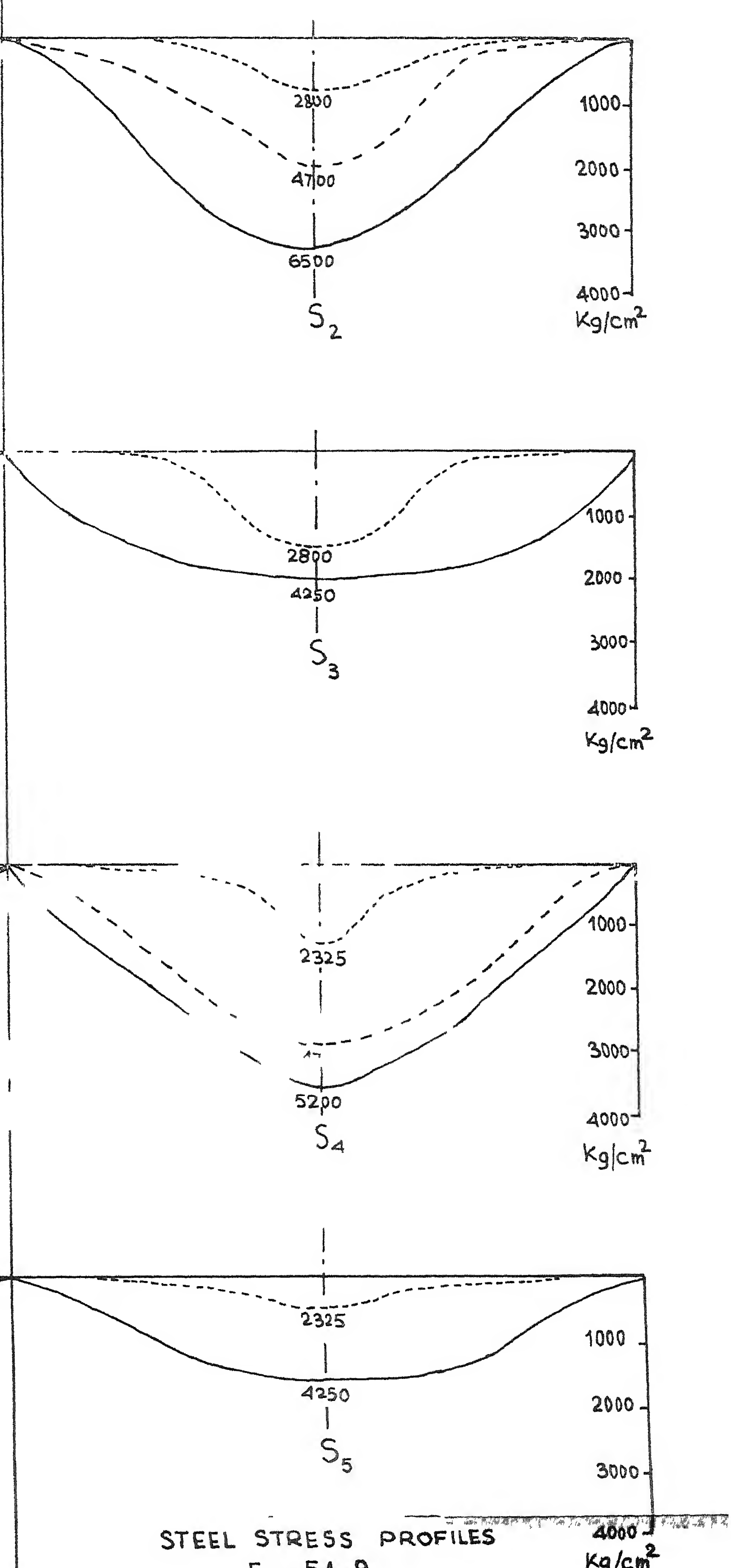


8000
7000
6000
5000
4000
3000
2000
1000
LOAD ( Kgs )
9-10
7-8
5-6
3-4
1-2
SLAB
$S_5$
2
4
6
8
10
12
14
16
18
20
22
24
DEFLECTION (mms)

Fig 5A.7




DISPLACEMENT PROFILES
Fig. 5A.8




STEEL STRESS PROFILES
Fig 5A.9

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